MATEMATISKA INSTITUTIONEN	Tentamensskrivning i
STOCKHOLMS UNIVERSITET	Combinatorics
Avd. Matematik	**Lösingar**
Examinator: Sven Raum	7.5 hp
	13th January 2020

1. Recursions and generating series (7 points)

   (a) Define the term **generating series**.

   (b) Assume that the generating series of a sequence $(a_n)_n$ of real numbers has positive radius of convergence and denote its generating function by $f = f(x)$. Prove that

   $$\frac{f(x)}{1-x}$$

   is the generating function of

   $$\left(\sum_{k\leq n} a_k\right)_n .$$

   (c) Find the generating function for the sequence $(n^3)_n$. You may freely use knowledge about the generating function for $(n^2)_n$.

   (d) Use the generating series methods to find the generating function $f = f(x)$ of the unique sequence $(a_n)_n$ satisfying

   $$a_n = 2a_{n-1} + n^3 \text{ for } n \geq 1 \qquad \text{and } a_0 = 1.$$

   Solution.

   (a) Given a sequence of numbers $(a_n)_{n\in\mathbb{N}}$, its generating series is the formal power series $\sum_{n\in\mathbb{N}} a_n x^n$.

   (b) Let $f(x) = \sum_{n\in\mathbb{N}} a_n x^n$ be the generating series of $(a_n)_n$, which by assumption has positive radius of convergence. We know that

   $$g(x) = \frac{1}{1-x} = \sum_{n\in\mathbb{N}} x^n$$

   for all $|x| < 1$. So $g(x)$ is the generating function of the constant sequence $(b_n)_n = (1)_n$. Since both power series defining f and g have positive radius of convergence, their formal product as power series equals the product of functions. So

   $$\frac{f(x)}{1-x} = f(x)g(x) = \sum_{n\in\mathbb{N}} \left(\sum_{k\leq n} a_k b_{n-k}\right) x^n = \sum_{n\in\mathbb{N}} \left(\sum_{k\leq n} a_k\right) x^n.$$

   This is what we had to show.

   (c) The generating function for $(n^2)_{n\in\mathbb{N}}$ is

   $$\frac{x(x+1)}{(1-x)^3} = \sum_{n\in\mathbb{N}} n^2 x^n.$$

Since this generating series has positive radius of convergence, its formal derivative equals its analytic derivative. So we obtain

$$\sum_{n=1}^{\infty} n^2 \cdot nx^{n-1} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{x(x+1)}{(1-x)^3}.$$

The right-hand side equals

$$\begin{aligned} \frac{(1+2x)(1-x)^3 - x(1+x)3(1-x)^2(-1)}{(1-x)^6} &= \frac{(1+2x)(1-x) + 3x(1+x)}{(1-x)^4} \\ &= \frac{1 - x + 2x - 2x^2 + 3x + 3x^2}{(1-x)^4} \\ &= \frac{1+4x+x^2}{(1-x)^4}. \end{aligned}$$

Multiplying this function by x, we hence obtain

$$\sum_{n \in \mathbb{N}} n^3 x^n = \frac{x(1+4x+x^2)}{(1-x)^4}.$$

(d) The generating series method assumes that the sequence $(a_n)_{n\in\mathbb{N}}$ has a generating function, say $f(x)$. For $n \geq 1$, we multiply the relation

$$a_n = 2a_{n-1} + n^3$$

with x^n and take the formal sum, in order to obtain the equality of power series

$$\sum_{n=1}^{\infty} a_n x^n = 2\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} n^3 x^n.$$

Making use the computed generating function for $(n^3)_{n\in\mathbb{N}}$, using the initial condition $a_0 = 1$ and substituting the generating function $f(x)$ for the generating series of $(a_n)_{n\in\mathbb{N}}$, we obtain

$$f(x) - 1 = 2x(f(x) + \frac{x(1+4x+x^2)}{(1-x)^4}.$$

Solving this expression for $f(x)$, we obtain

$$f(x) = \frac{x(1+4x+x^2)}{(1-x)^4(1-2x)} + \frac{1}{1-2x} = \frac{1 - 3x + 10x^2 - 3x^3 + x^4}{(1-x)^4(1-2x)}.$$

2. Graphs (7 points)

(a) Define the terms **directed graph** and **undirected graph**.

(b) Draw a planar depiction of the following graphs:

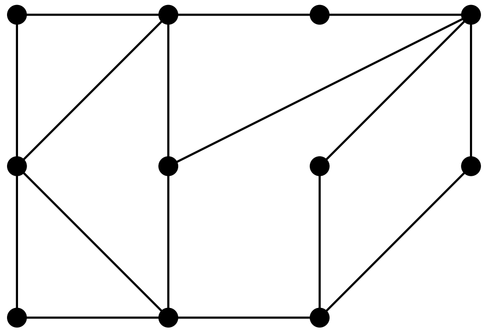
i. K_4.

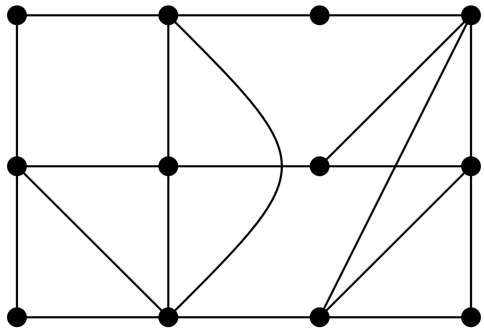
ii. $K_5 - e$ for an arbitrary edge $e \in E(K_5)$.

iii. $K_{3,2}$.

iv. $K_{3,3} - e$ for an arbitrary edge $e \in E(K_{3,3})$.

(c) Find an Euler circuit in each of the following graphs





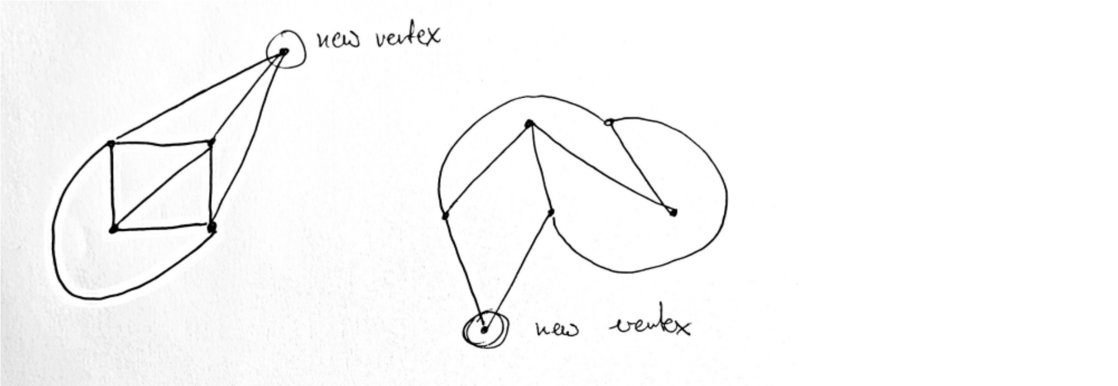
(d) Let G be a graph admitting an Euler circuit. Prove that $\deg(v)$ is even for all $v \in V(G)$.

(e) Calculate the chromatic polynomial of the n-cycle graph for all $n \in \mathbb{N}_{\geq 3}$.

Solution.

(a) A directed graph is a pair (V, E) of a non-empty set V and a subset $E \subset V \times V$. An undirected graph is a pair (V, E) of a non-empty set V and a subset $E \subset \{a \in \mathcal{P}(V) \mid |a| \in \{1, 2\}\}$, where $\mathcal{P}(V)$ denotes the set of all subsets of V.

(b) The following drawing indicates the additional vertex when passing from K_4 to $K_5 \setminus e$ and from $K_{3,2}$ to $K_{3,3} \setminus e$, respectively.



(c) Both graphs have a vertex of odd degree, so they do not admit any Euler circuit by the next item.

(d) Let $G = (V, E)$ be a graph admitting an Euler circuit. Since every loop of G contributes 2 to its adjacent vertex' degree, we may assume that G has no loops. Let $(v_1, \ldots, v_n)$ be an Euler circuit in G. Then for any $v \in V$, we find that

$$\deg(v) = |\{e \in E \mid v \in e\}| = |\{i \in \{1, \ldots, n\} \mid v \in \{v_i, v_{i+1(\text{mod} n)}\}\}|$$

is divisible by 2, since $v = v_i$ implies $v \in \{v_{i-1}, v_i\}$ and $v \in \{v_i, v_{i+1}\}$.

(e) We claim that $P(C_n, x) = (x-1)^n + (-1)^n(x-1)$ for all $n \in \mathbb{N}_{\geq 3}$. We will prove this by induction. For the case $n = 3$, we calculate the chromatic numbers

$$\chi_1(C_3) = 0$$
$$\chi_2(C_3) = 0$$
$$\chi_3(C_3) = 3! = 6$$

which leads us to the chromatic polynomial $P(C_3, x) = x(x-1)(x-2) = (x-1)^3 + (-1)^3(x-1)$. Let us next denote by L_n the path with n vertices. We know that

$$P(L_n, x) = x(x-1)^{n-1} \qquad n \geq 1.$$

This is relevant, since choosing any edge e of C_n, we have $C_n \setminus e = L_n$ as long as $n \geq 3$. Further, collapsing e, we obtain C_{n-1}. So the following formula holds for all $n \geq 3$: $P(C_n, x) = P(L_n, x) - P(C_{n-1}, x)$. We thus proceed by induction and assume that the result holds for some $n \geq 3$ and calculate

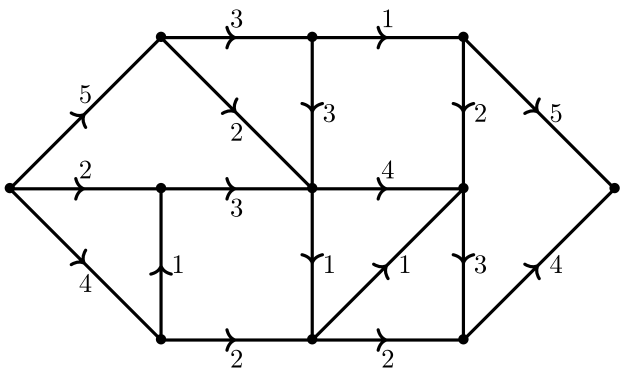
$$P(C_{n+1}, x) = x(x-1)^n - ((x-1)^n + (-1)^n(x-1)) = (x-1)^{n+1} + (-1)^{n+1}(x-1).$$

This completes the induction and hence the proof.

3. Networks (6 points)

(a) Define the term **flow** and the **value of a flow** on a transport network.

(b) Find a maximal flow and a minimal cut of the following transport network:



(c) Let $N = (G, c)$ be a transport network and $f : E(G) \to \mathbb{N}$ a flow on N. Show that for every cut (P, P^c) of N the following equality holds:

$$\text{val}(f) = \sum_{v \in P, w \in P^c} f(v, w) - f(w, v).$$

Solution.

(a) Given a transport network $N = (G, c)$, a flow on N is a function $f : V(G) \times V(G) \to \mathbb{N}$ such that

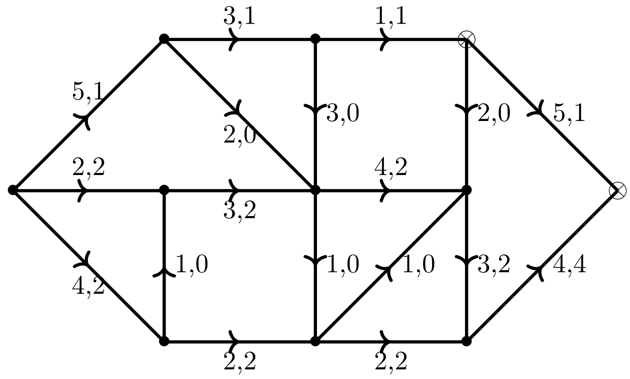
- $f(v, w) \leq c(v, w)$ for all $v, w \in V(G)$, and
- $\sum_{v \in V(G)} f(v, w) = \sum_{v \in V(G)} f(w, v)$ for all $w \in V(G)$ which are neither source nor sink of N.

The value of f is

$$\text{val}(f) = \sum_{v \in V(G)} f(a, v)$$

where a denotes the source of N.

(b) The following flow has value 5.



We find a cut with capacity 5 too. One such cut is (P, P^c) where P^c contains exactly the sink z and the unique adjacent vertex v such that (v, z) has capacity 5. The two vertices are marked in the graphic. By the max-flow-min-cut theorem, this already shows that the found flow is maximal and the indicated cut is minimal.

(c) We adopt the notation of the question and denote the source of N by a. Then

$$\begin{aligned}
\operatorname{val}(f) &= \sum_{v \in V(G)} f(a,v) && \text{(definition)} \\
&= \sum_{v \in V(G)} f(a,v) - f(v,a) && \text{(no incoming edges at the source)} \\
&= \sum_{v \in V(G)} f(a,v) - f(v,a) + \sum_{w \in P \setminus \{a\}} \sum_{v \in V(G)} f(w,v) - f(v,w) \\
& && \text{(equilibrium condition at non-terminal vertices)} \\
&= \sum_{\substack{w \in P \\ v \in V(G)}} f(w,v) - f(v,w) && \text{(simplification)} \\
&= \left(\sum_{\substack{w \in P \\ v \in P}} + \sum_{\substack{w \in P \\ v \in P^c}} \right) f(w,v) - \left(\sum_{\substack{w \in P \\ v \in P}} + \sum_{\substack{w \in P \\ v \in P^c}} \right) f(v,w) && \text{(splitting the sum)} \\
&= \sum_{\substack{w \in P \\ v \in P^c}} f(w,v) - \sum_{\substack{w \in P \\ v \in P^c}} f(v,w) && \text{(cancellation)} \\
&= \sum_{\substack{w \in P \\ v \in P^c}} f(w,v) - f(v,w).
\end{aligned}$$

This is what we had to show.

4. Algorithms (4 points)

(a) Define the terms **tree** and **spanning tree**.

(b) Describe how the depth-first algorithm starting at vertex $(0,0,0,0)$ runs on the 4-cube with the lexicographical ordering of vertices.

Solution.

(a) A tree is a connected, loop-free graph without cycles. Given a graph G, a spanning tree of G is a subgraph T of G that is a tree and satisfies $V(T) = V(G)$.

(b) Recall that the vertices of the 4-cube are 4-tuples $\{0,1\}^4$, which are adjacent if and only if they differ in exactly one coordinate. The lexicographical order on 4-tuples is given by $a > b$ if and only if $a \neq b$ and the first entry of a which differs from the respective entry of b is bigger. Formally, the latter condition can be described as $a_i > b_i$ for $i = \min\{j \in \{1,\ldots,4\} | a_j \neq b_j\}$. The depth-first algorithm then visits the following sequence of vertices, which defines a spanning tree (which is a path) of Q_4:

$$(0,0,0,0)$$
$$(0,0,0,1)$$
$$(0,0,1,1)$$
$$(0,0,1,0)$$
$$(0,1,1,0)$$
$$(0,1,0,0)$$
$$(0,1,0,1)$$
$$(0,1,1,1)$$
$$(1,1,1,1)$$
$$(1,0,1,1)$$
$$(1,0,0,1)$$
$$(1,0,0,0)$$
$$(1,0,1,0)$$
$$(1,1,1,0)$$
$$(1,1,0,0)$$
$$(1,1,0,1)$$

5. Finite geometry (6 points)

(a) Define the term **finite affine plane**.

(b) Define formally and illustrate with a graphic the examples of the affine planes of rank 2 and 3.

(c) Show that every finite affine plane admits at least three parallelity classes of lines.

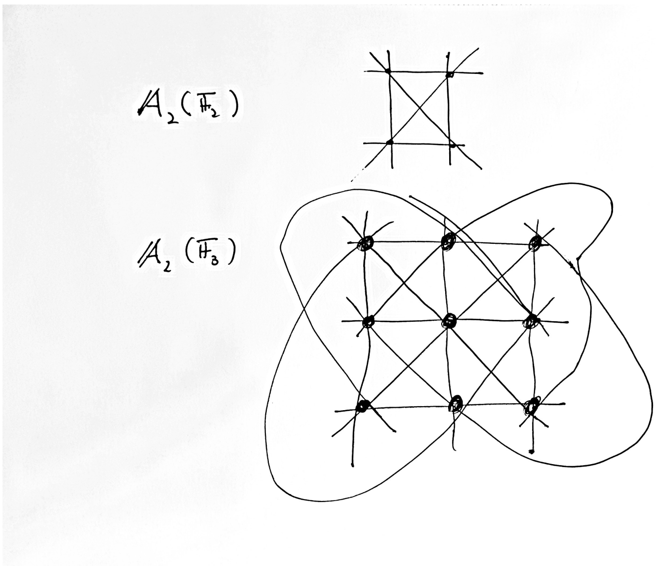
Solution.

(a) A finite affine place is a pair (P, L) of a set P and a subset $L \subset \mathcal{P}(P)$ such that
- for every pair of distinct points $p_1, p_2 \in P$ there is a unique $l \in L$ such that $p_1, p_2 \in l$,
- for every $l \in L$ and every $p \in P \setminus l$ there is a unique $l' \in L$ such that $p \in l'$ and $l \cap l' = \emptyset$, and
- there are points $p_1, \ldots, p_4 \in P$ such that for all $l \in L$ we have $|\{p_1, \ldots, p_4\} \cap l| \leq 2$.

(b) For a finite field k, we have $\mathbb{A}_2(k) = (k^2, L)$ where L consists of the lines

$$l_a = \{(x,y) \in k^2 \mid x = a\}$$
$$l_{a,b} = \{(x,y) \in k^2 \mid y = ax + b\}$$

for $a, b \in k$. Taking $k = \mathbb{F}_2$ and $k = \mathbb{F}_3$, we obtain finite affine planes of rank 2 and 3, respectively. They are illustrated by the following drawing.



(c) Let (P, L) be a finite affine plane and take $p_1, \dots, p_4$ such that for all $l \in L$ we have $|\{p_1, \dots, p_4\} \cap l| \leq 2$, whose existence is guaranteed by the definition of a finite affine plane. Denote by l_1, l_2, l_3 the lines through the pairs of points (p_1, p_4), (p_2, p_4) and (p_3, p_4), respectively. Then l_1, l_2, l_3 have pairwise non-empty intersection, but they are not equal thanks to the condition on $p_1, \dots, p_4$. They are hence from three different parallelity classes.