MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sven Raum

Tentamensomskrivning i
Combinatorics
Lösningar
7.5 hp
14th February 2020

1. Enumerative combinatorics (8 points)

   (a) Use the generalised inclusion-exclusion formula to calculate how many integers between 1 and 100 are divisible by exactly three different primes.

   (b) State and prove the pigeon hole principle.

   (c) Show that Euler's ϕ-function satisfies

   $$\phi(n) = n \cdot \prod_{\substack{p \text{ divides } n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right),$$

   for all $n \in \mathbb{N}_{\geq 1}$.

   Solution.

   (a) If conditions $c_1, \ldots, c_k$ are given on elements of a set S, and S_i denotes the number of elements satisfying at least i of these conditions, then the number of elements satisfying exactly k conditions is

   $$E_m = \sum_{i=0}^{k-m} (-1)^i \binom{m+i}{i} S_{m+i}.$$

   We apply this formula to $S = \{n \in \mathbb{N} \mid 1 \leq n \leq 100\}$ and the conditions $c_i(n)$ given by the statement that n is divisible by the i-th prime number. Note that the first four prime numbers are $2, 3, 5, 7$ and their product is 210, which is bigger than 100. So $S_i = 0$ for all $i \geq 4$. It hence follows that $E_3 = S_3$, which found to be equal to 8 after a systematic enumeration of all possible combinations.

   (b) This can be found in the lecture notes.

   (c) This can be found in the lecture notes.

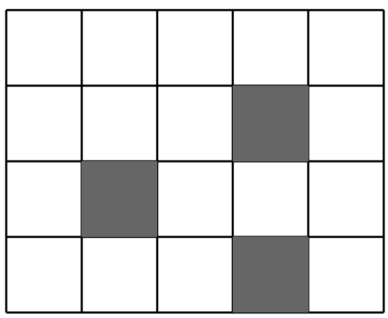
2. Rook polynomials (8 points)

   (a) Let us fix the following formalism for a combinatorial chessboard: a chessboard of size $m \times n$ is a matrix C of size $m \times n$ whose entries are either 0 or 1. We interpret an entry of C equal to 0 as a forbidden field, and an entry equal to 1 as an allowed field.

   Define the **rook numbers** and the **rook polynomial** of a combinatorial chessboard.

   (b) Draw all possible chessboards of size 2×2 and find their rook polynomials.

   (c) Calculate the rook polynomial of the following 4×5 chessboard.



(d) State formally and prove the fact that the rook polynomial is multiplicative.

Solution.

(a) Informally, the k-th rook number of a chessboard is the number of possible arrangements of k rooks on the allowed fields of the board, so that no two rooks attack each other. Formally, given a chessboard $C \in \mathrm{M}_{m,n}(\{0,1\})$, the k-th rook number of C is

$$r_k(C) = \big|\{(f_1, f_2) \mid f_1 : \{1,\dots,k\} \to \{1,\dots,m\} \text{ injective} \\ f_2 : \{1,\dots,k\} \to \{1,\dots,n\} \text{ injective, and} \\ C_{f_1(i),f_2(i)} = 1 \text{ for all } i \in \{1,\dots,k\}\}\big|.$$

Note that $r_k(C) = 0$ for all $k \geq \max\{m,n\}$. With this remark, it makes sense to define the rook polynomial of C as

$$r(C,x) = \sum_{k\in\mathbb{N}} r_k(C)x^k.$$

(b) Systematically listing all chessboards and a direct counting argument lead to a solution.

(c) Using the recursive formula $r(C,x) = r(C_e,x) + xr(C_s,x)$ several time, one arrives at the expression

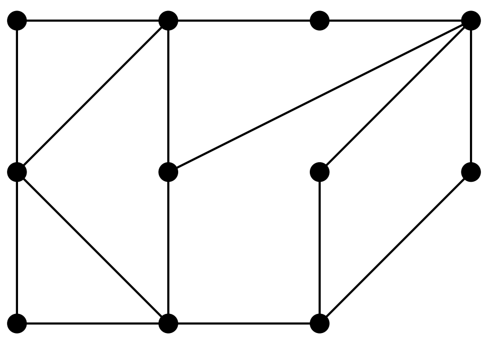
$$r(C,x) = 1 + 17x + 86x^2 + 144x^3 + 60x^4.$$

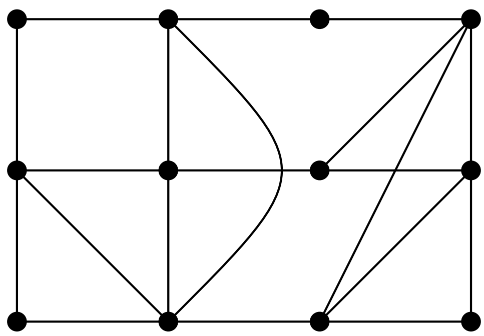
(d) This was a statement from the lecture.

3. Graphs (6 points)

(a) Define a **Hamiltonian cycle** in a graph.

(b) For each of the following graphs find a Hamiltonian cycle or show that there is none.





(c) Let G be a graph and $v, w \in V(G)$. Show that if $p = (v_1, \ldots, v_n)$ is a walk from v to w that has minimal length, then p is a path.

Solution.

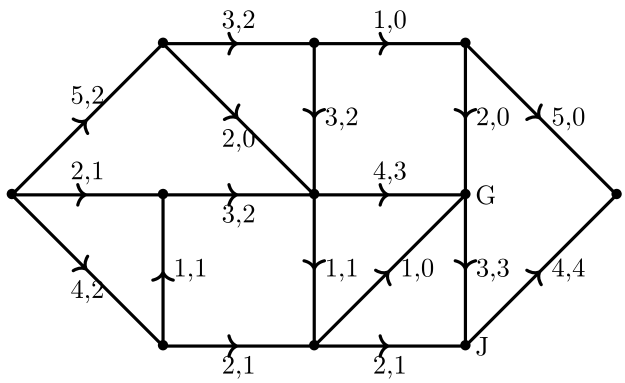
(a) Given a graph $G = (V, E)$ a Hamiltionian cycle in G is a cycle $(v_1, \ldots, v_n)$ such that $V = \{v_1, \ldots, v_n\}$ and $|V| = n - 1$.

(b) The first graph has a no Hamiltonian cycle, as can be shown by noticing the special role of vertices of degree 2. The second graph has a Hamiltonian cycle:

4. Networks (4 points)

(a) Let N be a transport network and f a flow on N. Define the term **f-augmenting path**.

(b) For the following flow, find all augmenting paths which are also paths in the underlying directed graph. Explain why you found all.



Solution.

(a) Given a transport network $N = (G, c)$ and a flow f on N, an f-augmenting path is a path $(v_1, \ldots, v_n)$ in the underlying undirected graph of G such that for all $i \in \{1, \ldots, n-1\}$ the following conditions are satisfied:

- $v_i \to v_{i+1}$ implies that $f(v_i, v_{i+1}) < c(v_i, v_{i+1})$, and
- $v_{i+1} \to v_i$ implies that $f(v_{i+1}, v_i) > 0$.

(b) There is a single such path.

5. Finite geometry (4 points)

(a) Define the term **parallel** in the context of finite affine planes.

(b) Show that being parallel defines an equivalence relation.

Solution.

(a) Given a finite affine plane (P, L), two lines $l_1, l_2 \in L$ are called parallel if either $l_1 = l_2$ or $l_1 \cap l_2 = \emptyset$.

(b) This is a statement from the lecture.