MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Lösningar till tentamensskrivning i
Matematik 3, kombinatorik , 7,5 hp
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1. a) The generating function for the number of partitions into parts of sizes 1, 2, and 3 is given by:

$$\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}=(1+x+x^2+x^3+x^4+...)(1+x^2+x^4+x^6+x^8+...)(1-x^3+x^6+x^9+x^{12}+...)$$

We are interested in the coefficient at x^{10} which equals 14.
Answer: 14

b) By duality (transposing the Young diagram) we conclude that there are equally many partitions of 10 into three parts as there are partitions of 10 into parts $1, 2, 3$, i.e., 14.

c) The generating function for the number of partitions with only odd parts is given by

$$\frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdot\frac{1}{1-x^9}\cdots$$

Since we are interested in the coefficient at x^{10} it is enough to take only the presented terms. Expanding the five geometric progressions and collecting the coefficients at x^{10} we get $10x^{10}$.
Answer: 10

d) Similarly, the generating function for the number of partitions with only even parts is given by

$$\frac{1}{1-x^2}\cdot\frac{1}{1-x^4}\cdot\frac{1}{1-x^6}\cdot\frac{1}{1-x^8}\cdot\frac{1}{1-x^{10}}\cdots$$

Since we are interested in the coefficient at x^{10} it is enough to take only the presented terms. Expanding the five geometric progressions and collecting the coefficients at x^{10} we get $7x^{10}$.
Answer: 7

2. Let a_n be the number of strings of length n containing only digits $0, 1, 2$ and having an odd number of even digits. Then there are $b_n = 3^n - a_n$ strings of length n with an even number of even digits. Consider all strings of length n satisfying our required condition and having the last digit even. Their number equals $2b_{n-1}$ since the last digit can be either 0 or 2. If we count our strings of length n for which the last digit is odd, then there are a_{n-1} such. Summarizing we get the recurrence

$$a_n = 2\cdot 3^{n-1} - a_{n-1},\ n \geq 2 \text{ with the initial condition } a_1 = 2.$$

General solution to the homogeneous equation $a_n = -a_{n-1}$ is given by $c(-1)^n$ where c is an arbitrary constant. A particular solution of the inhomogeneous recurrence relation can be found in the form $d\cdot 3^n$ where d is appropriately chosen. Substituting our ansatz in the equation gives $d = 1/2$. Finally, the initial condition $a_1 = 2$ gives $c = -1/2$.
Answer: $\frac{(-1)^{n+1}+3^n}{2}$.

3. The rook polynomial $r(C, x)$ starts with $1 + 12x$ and has degree at most 4. We need to count the number of rook placements with 2, 3 and 4 rooks.

In case of 2 rooks we get the following.

Placing rooks in rows 1 and 2, rows 1 and 3, rows 2 and 4, rows 3 and 4 gives 6 possibilities each.

Placing rooks in rows 1 and 4 gives 12 possibilities.

Placing rooks in rows 2 and 3 gives 2 possibilities.

Thus the coefficient at x^2 in $r(C, x)$ is equal to 38.

In case of 3 rooks:

Placing rooks in rows 1, 2, 3 and in rows 2, 3, 4 gives 4 possibilities each.

Placing rooks in rows 1, 2, 4 and in rows 1, 3, 4 gives 12 possibilities each.

Thus the coefficient at x^3 in $r(C, x)$ is equal to 32.

Finally there are 4 possibilities to place 4 rooks.

Answer: $r(C, x) = 1 + 12x + 38x^2 + 32x^3 + 4x^4$.

4. The recurrence relation defining the sequence $\{a_n\}_{n=1}^{\infty}$ is given by $a_n = a_{n-1} + a_{n-2}$ with the initial conditions $a_1 = 1, a_2 = 2$. Its characteristic equation equals $r^2 - r - 1 = 0$ whose roots are $\frac{1 \pm \sqrt{5}}{2}$. Thus the general solution of the above recurrence relation is

$$\alpha \left(\frac{1+\sqrt{5}}{2} \right)^n + \beta \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

The initial conditions give $\alpha = \frac{5+\sqrt{5}}{10}, \beta = \frac{5-\sqrt{5}}{10}$.

5. a) $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and $E = \{(a, b), (a, c), (b, c), (c, d), (c, e), (d, e)\}$. G has an Euler circuit $a - c - d - e - c - b - a$, but no Hamilton cycle since we can not pass the vertex c twice.

b) $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and $E = \{(a, b), (a, c), (a, d), (b, c), (c, d), (c, e), (d, e)\}$ has a Hamilton cycle $a - b - c - e - d - a$ but no Euler circuit since $\deg(a) = \deg(d) = 3$.

6. a) The minimal spanning tree has total weight 33 and consists of the edges:

$$(a, c); (c, f); (f, i); (f, h); (e, h); (e, g); (d, g); (b, e); (g, j).$$

b) The minimal path connecting a and j goes through $a - e - g - j$ and has total weight 16.

c) Maximal flow has total value 19 and its values on the edges are as follows. (There might exist other options.) $(a, b) \to 6$, $(a, c) \to 6$, $(a, e) \to 7$, $(b, d) \to 5$, $(b, e) \to 1$, $(c, e) \to 5$, $(c, f) \to 1$, $(d, g) \to 1$, $(d, h) \to 4$, $(e, g) \to 4$, $(e, h) \to 3$, $(e, i) \to 6$, $(f, h) \to 0$, $(f, i) \to 1$, $(g, j) \to 5$, $(h, j) \to 7$, $(i, j) \to 7$. Value $= 19$. Minimal cut of capacity 19 is given by $\{a, b, c, d, e, f, h, g\} \cup \{i, j\}$.