MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sofia Tirabassi

Exam in
Combinatorics
7.5 hp
January 12, 2023

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved. A score of at least 15 points will ensure a pass grade.

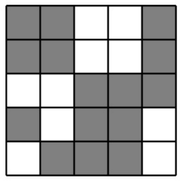
GOOD LUCK!

1. **Generating functions**

   (a) (2 points) Show that the exponential generating function of the sequence $a_n = (n+1)!$ is $f(x) = \frac{1}{(1-x)^2}$.

   (b) (3 points) Find the exponential generating function of $a_n = (n+1)!\, 2^{n-1}$.

2. **Rook polynomials:** Consider the following chessboard (only white cells are allowed)



   (a) (3 points) Compute the rook polynomial of the chessboard.

   (b) (1 point) What is the maximum number of rooks that can be placed?

   (c) (1 point) In how many ways can we place 3 rooks?

3. **Recursion:** (5 points) Suppose that we want to construct a n cm tall tower with red, blue and yellow blocks. The red blocks are 2 cm tall while the blue and yellow blocks are 1 cm tall. Let a_n be the number of ways to construct such a tower.

   (a) (1 point) Compute a_1 and a_2.

   (b) (2 points) Show that $a_n - 2a_{n-1} - a_{n-2} = 0$.

   (c) (2 points) Find a closed formula for a_n (you can take as definition $a_0 = 1$, if computations are to nasty with the boundary conditions from (a)).

4. **Graphs:** Given a graph $G = (V, E)$ its line graph LG is the graph whose set of vertices is E, and two distinct vertices are adjacent if and only if the corresponding edges have a vertex in common.

   (a) (1 point) Draw the line graph of $K_{1,3}$

   (b) (1 point) Given $e = \{v, w\} \in E$ compute $\deg(e)$ in LG as a function of $\deg(v)$ and $\deg(w)$.

   (c) (2 points) Compute the number of edges of LG (**Hint**: Let v be a vertex of degree d in G, how many pairs of edges does it contribute to the line graph?).

   (d) (2 points) Show that if G has an Euler circuit, then the same is true for LG. Is the converse of this statement true? Find a counterexample.

5. **Minimal spanning trees:** Consider the weighted graph in Figure 1

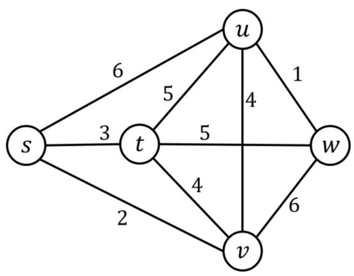


Figure 1: Find the spanning tree

(a) (2 points) Find a minimum spanning tree running Prim's algorithm starting from the vertex s. List the order in which vertices are added to the tree.

(b) (2 points) Find a minimum spanning tree using Kruskal's algorithm. List the order in which the vertex are added to the tree.

6. **Transport Networks:** Consider the transport network in Figure 2 (left) - where s is the source and t is the sink - and the initial flow f in Figure 2 (right).

(a) (1 point) Check that f is indeed a flow and compute its value.

(b) (3 points) Staring from the flow f, find a maximal flow for the network. Compute the value of the maximal flow.

(c) (1 point) Give the cut associated to the maximal flow you have found in (b), and check that this is indeed a minimal cut.

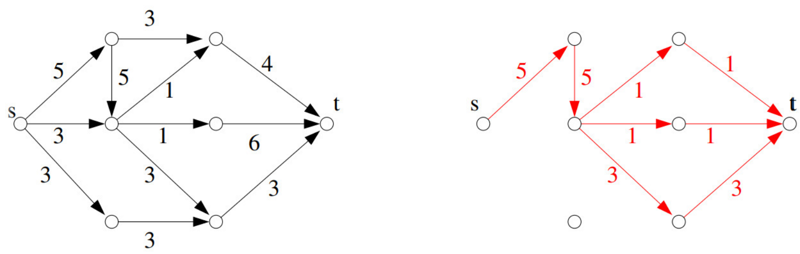


Figure 2: Network