MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sofia Tirabassi

Tentamensskrivning i
MM5023
2025-01-14

Instructions:

- During the exam you MAY NOT use textbooks, class notes, or any other supporting material.
- To solve a bullet point in the given exercise you can use all the preceeding points, even if you have not provided a solution for them.
- Start every problem on a new page, and write at the top of the page which problem it belongs to. (But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear and wrong argument, even if the final answer is correct.
- Write clearly and legibly.

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty

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(1) **Generating functions**
  (a) (1 pt) Find the generating function of the sequence $a_n = \sum_{k=0}^{n} 2k$
  (b) (2 pts) Show that $\frac{x+2x^2}{(1-x)^2}$ is the generating function of the sequence

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ 3(n-1)+1 & \text{if } n > 0. \end{cases}$$

  (c) (2 pts) Find the generating functions of the sequence

c_k = the sum of the first k positive integers n such that $n \equiv 1 (\mod 3)$

(2) **Rook Polynomial and Exclusion inclusion** A company has five employees A, B, C, D, and E who should be assigned 5 different task a, b, c, d and e, with the following constraint
- A is unsuited for tasks b and c
- B is unsuited for tasks a and c
- C is unsuited for tasks b and d and 2
- D is suited for all
- E is unsuited for d.

  (a) (1 pt) Draw a 5x5 chessboard with shaded cells corresponding to the "forbidden combinations"
  (b) (1 pts) Compute the rook polynomial of the chessboard made up by FORBIDDEN cells.
  (c) (3 pts) Compute the number of ways to assign the 5 jobs to the 5 employees.

(3) (4 pts) **Recursion** Solve the following recursion problem

$$a_n = 3a_{n-1} - 2a_{n-2} + 2^n$$

With initial conditions

$$a_0 = 1, \quad a_1 = 1$$

(4) **Graphs** Let p and q be two positve integer. A $p-q$ grid is a graph constructed in this way:

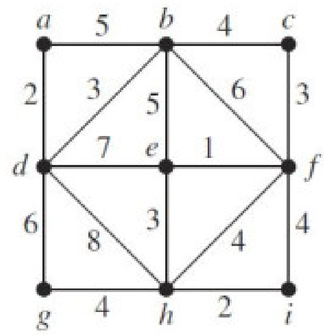

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FIGURE 1. Graph

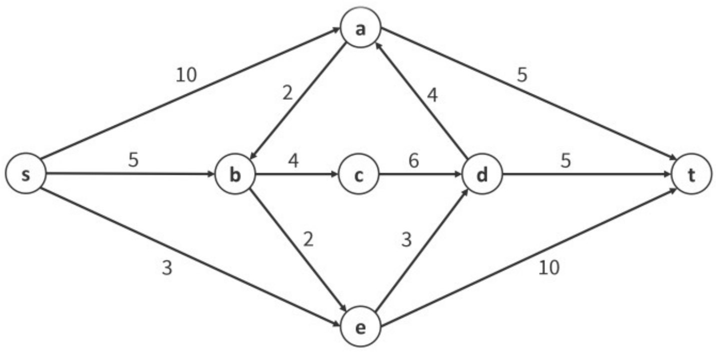

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FIGURE 2. Network

- the set of veritces is $V := \{1, \dots, p\} \times \{1, \dots, q\}$;
- two vertex (i, j) and (k, l) are adjacent if, and only if, $i = k$ and $|j - l| = 1$, or $j = l$ and $|i - k| = 1$.
  (a) (1 pt) Draw a 3-4 grid.
  (b) (2 pts) Determine the number of edges of a $p - q$ grid.
  (c) (1 pt) Decide if a for which p and q a $p - q$ grid admits an Euler path and an Euler circuit.
  (d) (2 pts) Determine the degree of of every vertex in a $p - q$ grid.
  (e) (2 pts) Give a conditions on the product pq that ensures that a $p - q$ grid has an Hamilton cycle

(5) (4 pts) **Minimal Spanning Tree** Consider the graph in figure 1. Find a minimal spanning tree by using either Kruskal's or Prim's algorithm. To get full point you have to declare which algroithm you are using and show all the iterations.

(6) (4 pts) **Max-flow Min-cut** Consider the Network in Figure 2. Provide a maximum flow for the network. Show that this is indeed a maximum flow by providing a cut with the same capacity.

GOOD LUCK!!!