

## HM5013 HT24 (Jouhari 4th)

(1) Generating functions.

(a) We have done this in class:

We start from the generating

function of  $b_n = 2$  for all  $n \geq 0$

$$\sum_{n=0}^{\infty} 2x^n = 2 \sum_{n=0}^{\infty} x^n = \frac{2}{1-x}$$

Now  $\frac{2x}{1-x}$  correspond to  $c_n = \begin{cases} 0 & n=0 \\ 2 & n \geq 1 \end{cases}$

Observe that  $a_n = \sum_{k=0}^n c_k$

Thus the generating function of

$$a_n \text{ is } \frac{1}{1-x} \cdot \frac{2x}{1-x} = \frac{2x}{(1-x)^2}$$

(b)  $a_n$  is the sum of the

sequence  $b_n = \begin{cases} 0 & \text{for } n=0 \\ 1 & \text{for } n=1 \\ 3 & \text{for } n \geq 1 \end{cases}$

the generating function of  $b_n$

is 
$$x + \frac{3x^2}{(1-x)} = \frac{x - x^2 + 3x^2}{(1-x)} = \frac{x + 2x^2}{(1-x)}$$

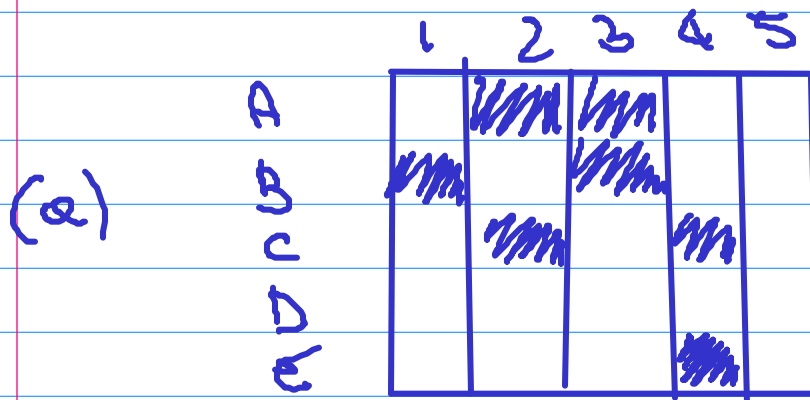
thus the generating function of  $a_n$  is

$$\left( x + \frac{3x^2}{(1-x)} \right) \frac{1}{1-x} = \frac{x + 2x^2}{(1-x)^2}$$

(c) 
$$c_k = \sum_{n=0}^k a_n$$

So has gf 
$$\left( x + \frac{3x^2}{(1-x)} \right) \frac{1}{1-x}$$

Problem 2 rook poly.



(b) Chessboard of forbidden cells

$$\text{Diagram} = C$$

$$r(C) = r(\text{Diagram 1}, x) + x r(\text{Diagram 2}, x)$$

$$= r(\text{Diagram 3}, x) + x r(\text{Diagram 4}, x)$$

$$+ x \left( r(\text{Diagram 5}) \cdot r(\text{Diagram 6}, x) \right)$$

$$= (1+x) r(\text{Diagram 7}, x) + x (1+x)(1+2x)$$

$$+ x \left( (1+x) \cdot (1+3x+x^2) \right)$$

$$= (1+x) (1+4x+4x^2) + x + 3x^2 + 2x^3$$

$$+ x (1+3x+x^2 + x+3x^2+2x^3)$$

$$= \underline{1} + \underline{4x} + \underline{4x^2} + \underline{x} + \underline{4x^2} + \underline{4x^3} + \underline{x}$$

$$\begin{aligned}
& + \underbrace{3x^2}_{6} + \underbrace{2x^3}_{6} + x(1 + 4x + 4x^2 + 2x^3) \\
& = 1 + 6x + 13x^2 + 8x^3 + x + 4x^2 + 4x^3 + 2x^4 \\
& = 1 + 7x + 15x^2 + 10x^3 + 2x^4
\end{aligned}$$

Let  $C_x$  be the condition

$X$  is given a wrong job

We want

$|C_A \cup C_E|$  that we can find

with inclusion exclusion

$$\sum_i |C_x| = 7 \cdot 4!$$

7 ways to assign 1 wrong job.  $4!$  to assign the other jobs

$$\sum_i |C_x \cap C_y| = 173!$$

7 ways to assign 2 wrong jobs and  $3!$  ways

to place the other

$$\sum_k |C_{x_{i_1}} \dots C_{x_{i_k}}| = r_k (\text{forbidden } C) \cdot (n-k)!$$

the # of ways to assign the jobs is

$$5! - 7(4!) + 17(3!) - 12(2!) + 2 = 25$$

(3) Recursion

Homogeneous prob

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda = 2, 1$$

General solution

$$a_n^{(h)} = A \cdot 2^n + B$$

For the non homogeneous prob

we have that that solutions

one of the form

$$a_n^{(p)} = Xn \cdot 2^n$$

$$Xn \cdot 2^n = 3X(n-1)2^{n-1} - 2X(n-2)2^{n-2} + 2^n$$

$$\begin{aligned} 4nX &= 6X(n-1) - 2X(n-2) + 4 \\ &= -6X + 4X + 4 \end{aligned}$$

$$2X = 4$$

$$X = 2$$

$$a_n^{(p)} = n2^{n+1}$$

General solution of uae has

then

$$a_n = A \cdot 2^n + B + n \cdot 2^{n+1}$$

$$1 = a_0 = A + B$$

$$1 = a_1 = 2A + B + 2$$

$$0 = A + 2$$

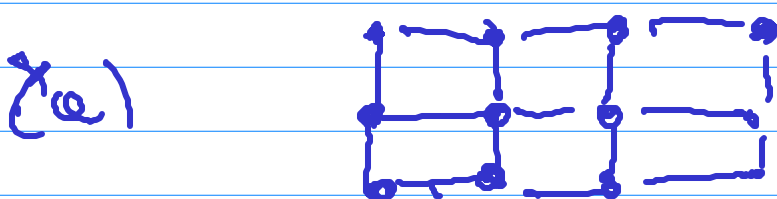
$$A = -2$$

$$B = 5$$

The final solution is

$$a_n = -2^{n+1} + 3 + n \cdot 2^{n+1}$$

### Problem 4



(b) there are  $q-1$  edges in every row and  $p$  rows  
 $p-1$  edges in cols and  $q$  cols

$$\begin{aligned} |\bar{E}| &= q(p-1) + p(q-1) \\ &= 2pq - p - q \end{aligned}$$

(c) let  $v = (1, a)$   $1 < a < q$

then there are 3 vertices adjacent to that

$$(1, a+1) \quad (1, a-1) \quad \text{and} \\ (2, a)$$

Similarly for  $(p, a)$

and  $(b, 1)$   $(b, q)$  for

$$1 < b < 2$$

Thus we need  $q \leq 3$   $p \leq 3$

and if  $q = 3$  then  $p = 2$

and if  $p = 3$  then  $q = 2$

in order for an Euler trail

for the circuit  $p, q \leq 2$

□

(c)  $(1, 1)$   $(1, q)$   $(p, 1)$   $(p, q)$

have degree 2

†  $(1, a)$   $(p, a)$   $(b, 1)$   $(b, q)$

have degree 3

all the other ~~are~~ have deg 4



(d) We know that

$$|E(G)| \geq \binom{|V(G)|-1}{2} + 2$$

ensures that  $G$  has a Hamiltonian  $C$ .

$$pq - p - q \geq \frac{(pq-1)(pq-2)}{2} + 2.$$

### Problem 5

Prism with input  $\{a\}$

$$P = \{a\}$$

$$P = \{a, d\}$$

$$P = \{a, d, b\}$$

$$P = \{a, d, b, c\}$$

$$P = \{a, d, b, c, f\}$$

$$P = \{a, d, b, c, f, e\}$$

$$P = \{a, d, b, c, f, e, h\}$$

$$P = \{a, d, b, c, f, e, h, i\}$$

$$P = \{a, \text{---}, i, g\}$$

Kruskal {ef}

{ef} {ad}

{ef} {ad} {hi}

{ef} {ad} {hi} {cf}

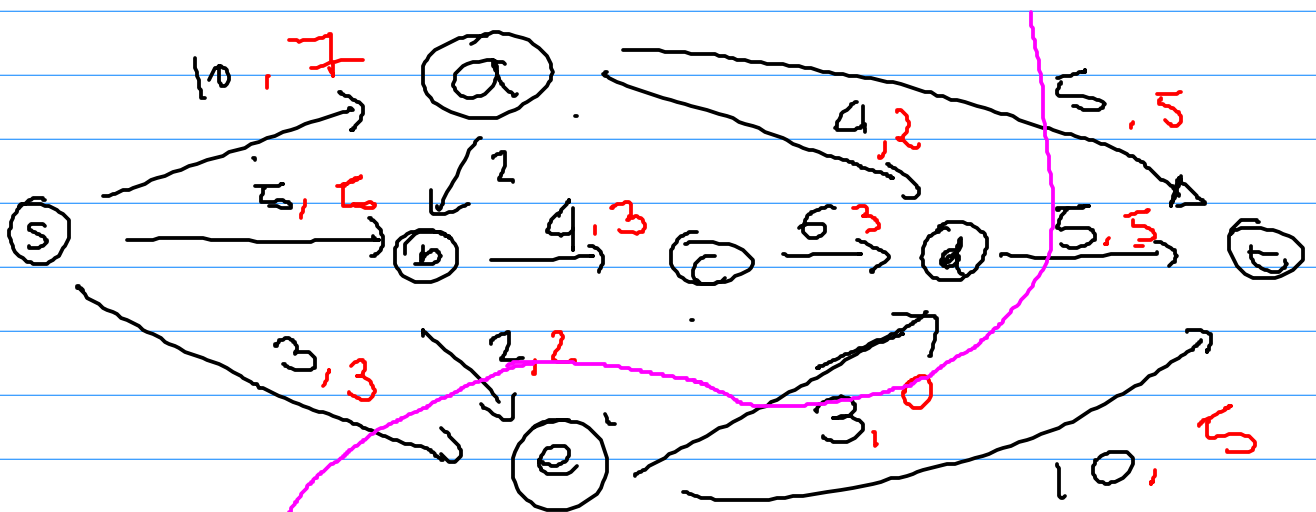
\_\_\_\_\_ {eh}

\_\_\_\_\_ {db}

\_\_\_\_\_ {cb}

\_\_\_\_\_ {gh}

Problem 6



$$\text{Value} = 6 + 7 + 3 = 15$$

The cut  $P = \{s, a, b, c, d\}$

The capacity of the cut

$$C(a,e) + C(d,t) + C(b,e) + C(a,t)$$

$$= 3 + 5 + 2 + 5 = 15$$

the capacity of the cut and  
value of the flow match.