MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sofia Tirabassi

Tentamensskrivning i
MM5023
2025-02-20

Instructions:

- During the exam, you may not use textbooks, class notes, or any other supporting material.
- To solve a bullet point in the given exercise, you can use all the preceding points, even if you have not provided a solution for them.
- Start every problem on a new page and write at the top of the page to which problem it belongs. (But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for an unclear or wrong argument, even if the final answer is correct.
- Write clearly and legibly.

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty. This exam is written in THREE PAGES

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(1) **Generating functions**

(a) (2 pt) Let $f(x)$ be the exponential generating function of the sequence a_0, a_1 , ... Show that $f'(x)$ is the exponential generating function of the sequence a_1, a_2, ...

(b) (3 pts) Let a_n denote the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = n$$

where $x_1 \geq 0$, $x_2 \geq 3$, $2 \leq x_3 \leq 5$ and $1 \leq x_4 \leq 5$. Compute the generating function of a_n.

(2) **Rook Polynomials** The BISHOP is the chess piece that can move diagonally. Given a chessboard C one can define the *bishop polynomial*

$$B(C, x) = \sum b_i x^i$$

where b_i is the number of ways of placing i non-attacking bishops on C

(a) (1 pt) Compute the bishop polynomials of a 2×2 chessboard.

Given a $n \times m$ chessboard C, without forbidden positions, suppose to color its cells by alternating two different colors (white and black). Let R_w be the chessboard obtained by C by rearranging the white cells in such a way that every top-left diagonal is a row, and every top-right diagonal is a column. Denote by R_b the chessboard obtained similarly from the black cells. In the figure 1 you can see an example.

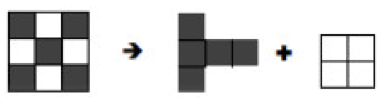

FIGURE 1. Chessboard

(b) (1 pt) Decompose a 4×4 chessboard in R_w and R_b.

(c) (2 pts) Show that, given a $n \times m$ chessboard C without forbidden places,

$$B(C, x) = r(R_b, x) \cdot r(R_w, x)$$

where $r(R_b, x)$ denotes the rook polynomial of R_b and, similarly, $r(R_w, x)$ stands for the rook polynomial of R_w.

(d) (3 pts) Compute the Bishop polynomial for a 4×4 chessboard.

(3) (4 pts) **Recursion** Solve the following recursion problem

$$a_n = 3a_{n-1} - 2a_{n-2} + 300$$

With initial conditions

$$a_0 = 1, \quad a_1 = 1$$

(4) **Graphs** Consider the graph in figure 2

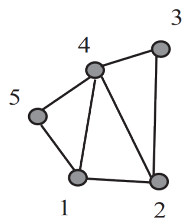

FIGURE 2. Graph

(a) (1 pt) Determine if it admits an Euler circuit or trail.
(b) (3 pts) Compute the chromatic polynomial of the graph.
(c) (1 pt) Compute the chromatic number of the graph.

(5) (5 pts) **Shortest path algorithm** Consider the graph in figure 3. Use Dijkstra's algorithm to compute all shortest paths starting at node s. To get full points you have to show every iteration, listing the value of the previous vertices and the length of the path. It is ok to use a table.

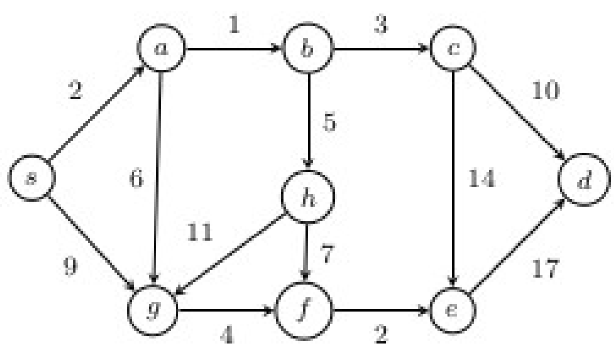

FIGURE 3. Graph

(6) (4 pts) **Max-flow Min-cut** Consider the Network in Figure 4. Provide a maximum flow for the network. Show that this is indeed a maximum flow by providing a cut with the same capacity.

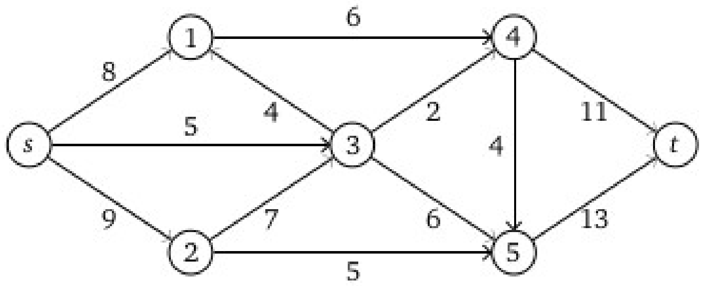

FIGURE 4. Network

GOOD LUCK!!!