MM5023 exam 20/02/2025

Exercise 1

(a) $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ is the generating function of (a_n). In particular it converges absolutely $\forall\, |a| < \rho$ and we can compute
& uniformly

$$f'(x) = \sum_{n} \frac{d}{dx}\left(\frac{a_n}{n!} x^n\right) = \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} x^{n-1}$$

$$= \sum_{k=0}^{\infty} \frac{a_{k+1}}{k!} x^k$$

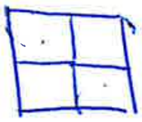
this is the exponential generating function of $(a_{k+1}) = a_1, a_2 \cdots$

(b)

$$f(x) = \underbrace{\frac{1}{1-x}}_{X_1} \; \underbrace{\frac{x^3}{1-x}}_{X_2} \; \underbrace{(x^2+x^3+x^4+x^5)}_{X_3} \underbrace{(x+x^2+x^3+x^4+x^5)}_{X_4}$$

Exercise 2

(a)



For 1 bishop we have 4 possibilities

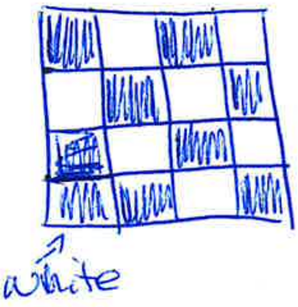
For 2 bishop we have 2 possibilities for each bishop

$\Rightarrow 4$

$$\boxed{1+4x+4x^2}$$

Note that we cannot place more than 2 bishops

(b)



(c) Firs note that no bishop on white cells can overtake a bishops on white cells & conversely no bishop on white cell can overtake a Bishoup on black cell.

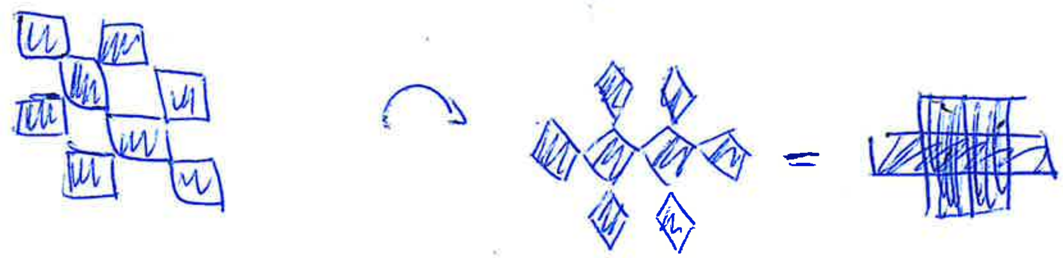
So the ~~only~~ white & black cell give disjoint chessbord and

$$B(C,x) = B(C_w,x)\,B(C_b,x)$$

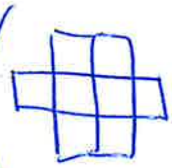
as in the rook polynomial case.

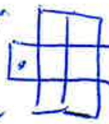
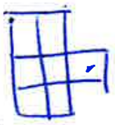
Now we want to see that $B(C_w,x)$ and $B(C_b,x)$ can be computed has rook polynomials.

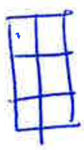
Observe that R_w is obtained by C_w by making diagonals into row and column. Thus a bishop placed in a cell of B_w will attack the same cases as a rook placed in the corresponding cell of R_w. And similarly for ~~whit~~ black. A visual way to show this is by rotation

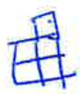
$C_b =$ 

(b) from (c)

$$B(C,x) = r\left(\right.$$  $$\left. , x\right)^2$$

$r($  $, x) = r($  $, x) + x\, r($  $, x)$

$= r($  $, x) + 2x\, r($  $x)$

$= r($  $, x) + x\, r($  $, x)$

$$+ 2x\left(1+4x+2x^2\right)$$

$$= 1+4x+2x^2 + 2x(1+2x) + 2x + 8x^2 + 8x^3$$

$$= 1 + 4x + 2x^2 + 2x + 4x^2 + 2x + 8x^2 + 8x^3$$

$$= 1 + 8x + 14x^2 + 8x^3$$

$$\Rightarrow \quad B(C, x) = (1 + 8x + 14x^2 + 8x^3)^2$$

Exercise 3

$$x^2 - 3x + 2 = 0 \qquad (x-2)(x-1) = 0$$

$$x = 2, 1$$

$$a_n^{(h)} = A 2^n + B$$

$a_n^{(p)}$ we choose $\alpha n = a_n^{(p)}$

$$\alpha n = 3\alpha(n-1) - 2\alpha(n-2) + 300$$

$$0 = -3\alpha + 4\alpha + 300$$

$$\alpha = -300$$

$$a_n = A \cdot 2^n + B - 300n$$

find A and B

$$\begin{cases} 1 = A + B \\ 1 = 2A + B - 300 \end{cases}$$

$$\begin{cases} A + B = 1 \\ 2A + B = 301 \end{cases}$$

$$A = 300$$
$$B = -299$$

$$\boxed{a_n = 300 \cdot 2^n - 299 + 300n}$$

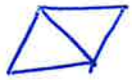
Exercise 4

(a) no circuit since vertices 1 & 2 have odd degree but it admits a trail since there are just two vertices of odd degree

(b)

 $= G$

we see that $G = G_1 \cup G_2$

$G_i =$  $\qquad G_1 \cap G_2 = \triangle = K_3$

$$P(G,\lambda) = \frac{P(G_1\,\lambda)^2}{P(\triangle\,\lambda)} = \frac{P(G_1\,\lambda)^2}{\lambda(\lambda-1)(\lambda-2)}$$

$P(G_1\,\lambda)$:

$G_1 = K_4 -$ diagonal edge.

$P(G_1\,\lambda) = P(K_4\,\lambda) + P(K_3\,\lambda)$

where K_3 is K_4 with a collapsed diagonal edge.

$= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$

$= \lambda(\lambda-1)(\lambda-2)^2$

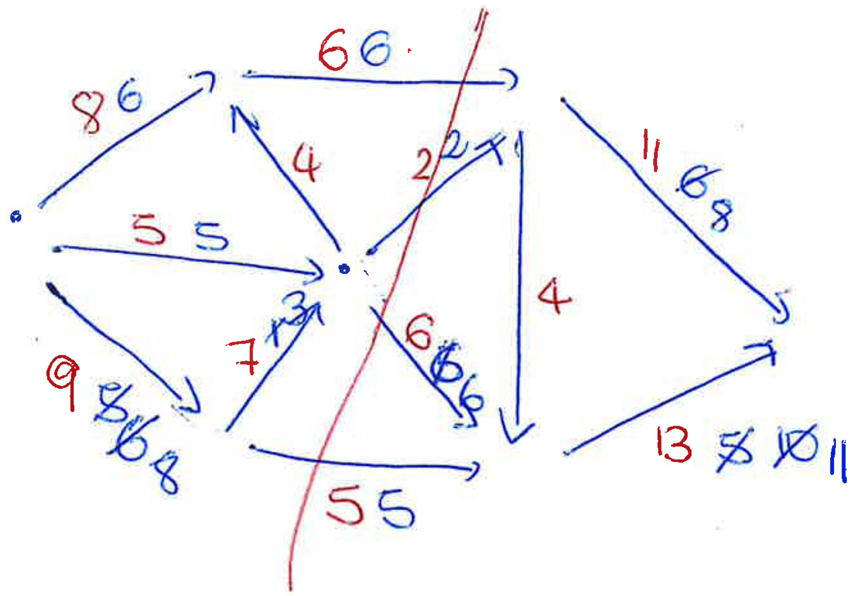
$P(G\,\lambda) = \lambda(\lambda-1)(\lambda-2)^3$

(c) $\chi(G) = 3$ since $P(G,2) = 0$ but $P(G,3) \neq 0$

Exercise 5

s	a	b	c	d	e	f	g	h
-	∞	∞	∞	∞	∞	∞	∞	∞
-	$\boxed{S,2}$	∞	∞	∞	∞	∞	S9	∞
-	S,2	$\boxed{a3}$	∞	∞	∞	∞	a8	∞
-	S2	a,3	$\boxed{b6}$	∞	∞	∞	a8	b8
-	S2	a,3	b6	c,16	c,20	∞	a8	$\boxed{b8}$
-	S2	a,3	b6	c16	c,20	d15	$\boxed{a,8}$	b8
-	S2	a3	b6	c16	c20	$\boxed{d12}$	a,8	b8
-	S2	a3	b6	c16	$\boxed{c14}$	d12	a8	b8
-	S2	a3	b6	$\boxed{c16}$	c14	d12	a8	b8

Exercise 6



$\Delta p = 6$

$\Delta p = 5$

$\Delta p = 5$

$\Delta p = 1$

$\Delta p = 2$

The value of the flow is

$$6+5+8 = 19 = 8+11$$

$P = \{S, 1, 3, 2\}$

$$C(P, P^c) = C(14) + C(34) + C(3S) + C(2S)$$
$$= 6+2+6+5 = 19$$