MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sofia Tirabassi

Tentamensskrivning i
MM5023
2026-01-14

Instructions:

- During the exam you MAY NOT use textbooks, class notes, or any other supporting material.
- To solve a bullet point in the given exercise you can use all the preceeding points, even if you have not provided a solution for them.
- Start every problem on a new page, and write at the top of the page which problem it belongs to. (But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear and wrong argument, even if the final answer is correct.
- Write clearly and legibly.

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty

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(1) **Generating functions and Recursion** The Tower of Hanoi is a puzzle consisting of three rods and a number of disks of various diameters, which can slide onto any rod. The puzzle begins with the disks stacked on one rod in order of decreasing size, the smallest at the top, thus approximating a conical shape. The objective of the puzzle is to move the entire stack to one of the other rods, obeying the following rules:
- Only one disk may be moved at a time.
- Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty rod
- No disk may be placed on top of a disk that is smaller than it.

In Figure 4 you see an example with 3 disks. Let a_n be the number of

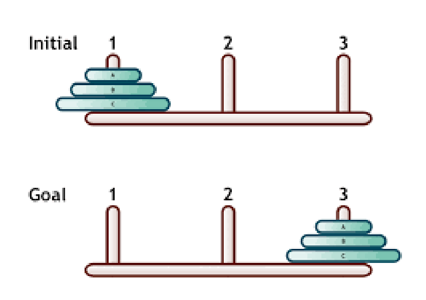

FIGURE 1. Tower of Hanoi

moves that you need to complete the puzzle with n disks.

(a) (2 pt) Show that $a_n = 2a_{n-1} + 1$ for $n \geq 2$ and $a_1 = 1$.

(b) (3 pts) Show that the generating function for a_n is $f(x) = \frac{x}{(1-x)(1-2x)}$.

(c) (2 pts) Give a non recursive formula for a_n. You can use whatever methods you want and you can use without proof that

$$f(x) = \frac{1}{1-2x} - \frac{1}{1-x}.$$

(2) **Rook Polynomial and Exclusion inclusion** Consider the chessboard in Figure 2.

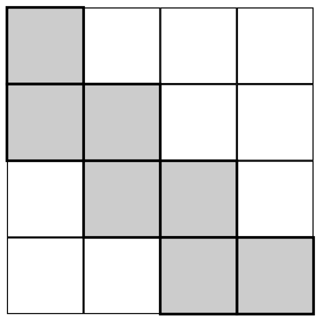

FIGURE 2. Chessboard

(a) (3 pt) Compute the rook polynomial of the chessboard consisting of SHADED cells.
(b) (3 pts) Let S_4 be the set (group) of bijective functions $\sigma : \{1, \dots, 4\} \to \{1, \dots, 4\}$. Count how many such functions exist such that $\sigma(i) \neq i, i+1$.

(3) **Graphs**
(a) (3 pts) Show that every tree is a bipartite graph.
(b) (3 pts) Show that every tree is planar.
(c) (3 pts) Show that the chromatic polynomial of a tree with n vertices is

$$\lambda(\lambda - 1)^{n-1}$$

(4) (4 pts) **Spanning trees** Consider the graph in Figure 3. Find a minimal spanning tree by using either Kruskal's or Prim's algorithm. To get full point you have to declare which algorithm you are using and show all the iterations.

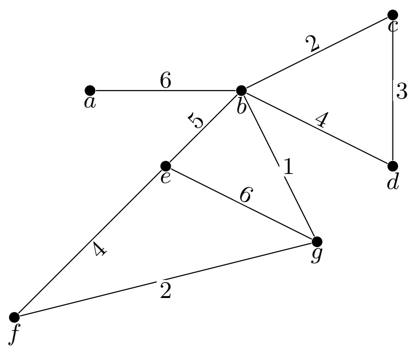

FIGURE 3. Graph

(5) (4 pts) **Max-flow Min-cut** Consider the Network in Figure 4. Provide a maximum flow for the network. Show that this is indeed a maximum flow by providing a cut with the same capacity.

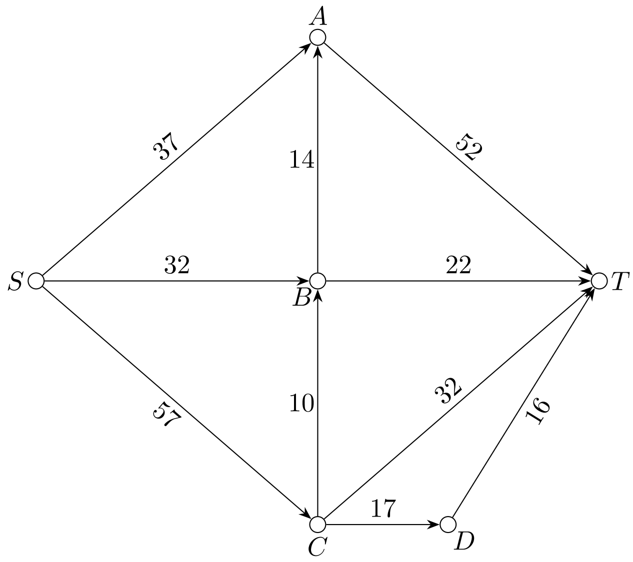

FIGURE 4. Network

GOOD LUCK!!!