MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensomskrivning i
Combinatorics
7.5 hp
11th January 2022

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved. A score of at least 15 points will ensure a pass grade.

GOOD LUCK!

1. **Partitions:** (2 points)
Consider r a positive integer and let a_r be the number of (unordered) partitions of r such that:

 - no summand is larger than 4;
 - 3 appears at least 3 times;
 - 4 appears at most 2 times.

Express of the generating function of a_r, $r \in \mathbb{N}_{>0}$, as a quotient of polynomials. **Solution:** We know from the lectures and textbook that the generating function for $p(r)$ the number of unordered partitions of r is

$$\prod_{n \in \mathbb{N}_{>0}} \frac{1}{1-x^n}.$$

If we assume that the highest summand is 4 the generating functions becomes

$$\prod_{n=1}^{4} \frac{1}{1-x^n}.$$

The conditions that there could be at most 2 fours change the generating function in the following

$$\prod_{n=1}^{3} \frac{1}{1-x^n} \cdot (1+x^4+x^8).$$

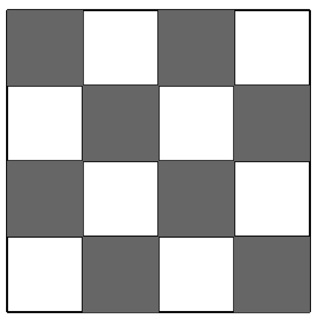
We just have to see the effect of the condition on the numbers of 3. So we have that

$$\begin{aligned} f(x) &= \left(\frac{1}{1-x}\right) \cdot \left(\frac{1}{1-x^2}\right) \cdot \left(\sum_{n=3}^{\infty} x^{3n}\right) \cdot (1+x^4+x^8) \\ &= \left(\frac{1}{1-x}\right) \cdot \left(\frac{1}{1-x^2}\right) \cdot \left[x^9 \cdot \left(\sum_{n=1}^{\infty} x^{3n}\right)\right] \cdot (1+x^4+x^8) \\ &= \left(\frac{1}{1-x}\right) \cdot \left(\frac{1}{1-x^2}\right) \cdot \left(\frac{x^9}{1-x^3}\right) \cdot (1+x^4+x^8) \\ &= \left(\frac{1}{1-x}\right) \cdot \left(\frac{1}{1-x^2}\right) \cdot \left(\frac{1}{1-x^3}\right) \cdot (x^9+x^{13}+x^{17}) \end{aligned}$$

2. **Rook polynomials:**

   (a) (2 point) Define the **rook numbers** and the **rook polynomial** of a chessboard C.

   (b) (3 points) Calculate the rook polynomial of the following 4×4 chessboard.



   (c) (2 point) State formally how the rook polynomial of the union of two disjoint chessboards C_1 and C_2 can be written in terms of the rook polynomials of the C_i's

(d) (2 points) Prove your statement in point (d).

Soulution: For (a), (b), and (c) we refer to the textbook.
(d) Using the forumla $r(C,x) = xr(C_e,x) + r(C_s,x)$ one arrives at the result

$$r(C,x) = 1 + 8x + 20x^2 + 16x^3 + 4x^4$$

3. **Recursion:**
Consider the following recursion relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 5$$

With boundary conditions $a_0 = 0$ and $a_1 = 1$.

(a) (3 points) Solve the relation finding a closed formula for a_n.

(b) (2 points) Express the generating function of the sequence $\{a_n\}_{n\in\mathbb{N}}$ as a quotient of polynomials.

Solution: (a) The characteristic equation of the recursion relation is $x^2 - 6x + 9 = 0$ which has a double real root $x = 3$. Therefore the general solution of the homogeneous relation is

$$a_n^{(h)} = (A + Bn)3^n,$$

with A and B real numbers. We observe that the right hand side is of the form $5(1)^n$. Has 1 is not a root of the characteristic equation, a particular solution of the recursion relation will be

$$a_n^{(p)} = \alpha \tag{1}$$

for some constant α. We plug this in the recursion relation to determine α. We get

$$\alpha - 6\alpha + 9\alpha = 5$$

from which we deduce that $\alpha = \frac{5}{4}$. So the general solution of the recursion relation is

$$a_n = (A + Bn)3^n + \frac{5}{4}.$$

Now we have to determine the values of A and B using the boundary conditions. We get that

$$0 = a_0 = A + \frac{5}{4},$$

thus we have that $A = -\frac{5}{4}$. From the second condition we have that

$$1 = a_1 = (A + B)3 + \frac{5}{4} = -\frac{5}{2} + 3B.$$

We deduce that $B = -\frac{7}{6}$. Thus the solution of the recursion relation with boundary condition is

$$a_n = (-\frac{5}{4} + \frac{7}{6}n)3^n + \frac{5}{4}.$$

(b) We have that

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 6\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 9\sum_{n=0}^{\infty} a_n x^{n+2} = 5\sum_{n=0}^{\infty} x^{n+2}.$$

We can rewrite this has

$$f(x) - a_0 - a_1 x - 6x(f(x) - a_0) + 9x^2 f(x) = 5\frac{x^2}{1-x},$$

where f is the generating function of the a_n's. Now we use the boundary conditions an s we get the following equation

$$f(x) - x - 6xf(x) + 9x^2 f(x) = 5\frac{x^2}{1-x},$$

which we solve for $f(x)$ and get

$$f(x) = \frac{4x^2 + x}{(1-x)(1-6x+9x^2)}.$$

4. **Graphs:**
Consider the (simple and loop-free) complete bipartite graph $K_{n,m}$.
   (a) (2 points) Give conditions on n and m such that $K_{n,m}$ is connected.
   (b) (2 points) Give conditions on n and m such that $K_{n,m}$ has an Euler circuit.
   (c) (2 points) Give conditions on n and m such that $K_{n,m}$ has an Hamilton path.
   (d) (2 points) Compute the chromatic polynomial of $K_{2,2}$. (**Formula**: you can use that $p(K_n, x) = x(x-1)(x-2)\cdots(x-n+1)$)

Solution: (a) If n and m are both positive, the graph is connected. In fact let consider $V(K_{n,m}) = V_1 \cup V_2$, and take a and b in V. If they belong to different V_i's then the edge $\{a, b\}$ is in $E(K_{n,m})$. Suppose otherwise that both a and b are in V_1 then there is a vertex $c \in V_2$ (m is positive) which is adjacent to both a and b (by the completeness of $K_{n,m}$. Thus (a, c, b) is a path connecting a and b. A similar argument, with the assumption that n is positive let us construct a path from a to b when a and b are in V_i.
(b) To have an Euler circuit the degree of every vertex has to be even. Let $a \in V(K_{n,m}) = V_1 \cup V_2$. Then $\deg(a) = m$ if $a \in V_1$ or n otherwise. Therefore $K_{n,m}$ has an Euler circuit if, and only if, both n and m are even.
(c) If the degree of two non adjacent vertices is bigger or equal $n + m \geq 3$ we have an Hamilton path. Two vertices a and b in $V(K_{n,m}) = V_1 \cup V_2$ are not adjacent if, and only if, they belong to the same V_i. In this case their degree is the size of the other V_i. Thus we have that $2m \geq m+n$ and $2n \geq n+m$ are two conditions that ensure the existence of an Hamilton path. We deduce that an Hamilton path exists when $n = m \geq 2$.
Alternatively, we have an Hamilton path if $n + m \geq 3$ and the sum of the degrees of any two vertices is at least $nm - 1$. We get this 3 conditions

   (a) $n + m \geq n + m - 1$, which is always satisfied
   (b) $2n \geq n + m - 1$, which is equivalent to $n \geq m - 1$;
   (c) $2m \geq n + m - 1$ which is equivalent to $m \geq n - 1$.

Thus if we take $n = m - 1$, or $m = n - 1$, all the conditions are satisfied and there is an Hamilton path. (d) This is an example in the book.

5. **Latin squares:**
Let q be a prime different from 2 or 3. Define the $q \times q$ matrix $A = (a_{ij})$ by $a_{ij} \equiv 2i + j (\mod q)$
   (a) (2 points) Write A when $q = 5$. Observe that it is a Latin square.
   (b) (2 points) For $q = 5$ find a Latin square which is orthogonal to A. (**Hint:** 3 is a unit in $\mathbb{F}_5$)
   (c) (2 points) Show that for every q, the matrix A is a Latin square. (**Hint:** You need to show that $a_{ij} = a_{ik}$ implies $j = k$ and that $a_{ij} = a_{lj}$ implies $i = l$.)

Solution: (a)

0	1	2	3	4
2	3	4	0	1
4	0	1	2	3
1	2	3	4	0
3	4	0	1	2

(b) We know that $a_{i,j} \equiv 3i + j (\mod q)$ will produce a Latin square ortogonal to the one given above:

0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1

(c) Suppose that $a_{ij} = a_{ik}$. Then we have that

$$2i + j \equiv 2i + k (\mod q).$$

The rules of the operation in $\mathbb{F}_q$ guarantee that $j = k$. Suppose now that $a_{ij} = a_{lj}$ for some j then we have

$$2i + j \equiv 2l + j (\mod q),$$

which is equivalent to $2i \equiv 2j (\mod q)$. Since 2 is a unit in $\mathbb{F}_q$ we deduce that $i \equiv j (\mod q)$.