MATEMATISKA INSTITUTIONEN	Exam in
STOCKHOLMS UNIVERSITET	Combinatorics
Avd. Matematik	7.5 hp
Examinator: Sofia Tirabassi	February 8th, 2023

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved. A score of at least 15 points will ensure a pass grade.

GOOD LUCK!

1. **Generating functions** Let C_n be the sequence of Catalan numbers, that is numbers defined by the relation

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1},$$

With initial conditions $C_0 = C_1 = 1$.

   (a) (1 point) Compute C_2 and C_3.

   (b) (2 points) Let $G(x)$ be the generating function of the sequence C_n, show that $x(G(x))^2 - G(x) + 1 = 0$.

   (c) (2 points) Conclude (using the initial conditions) that

$$G(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

   You can use that the McLaurin polynomial of $\sqrt{1-4x}$ is

$$p(x) = 1 - 2x - 2x^2 + \cdots +$$

   (**Hint**: For G to be a generating function it has to be bounded when x is close to 0)

2. **Rook polynomials:** (5 points) Compute the rook polynomial of the the chessboard C obtained from a 4×4 chessboard by forbidding all the cells below on of the diagonals.

3. **Recursion:** (4 points) Consider the following recursion relation

$$a_{n+2} + 4a_{n+1} + 4a_n = 3(-2)^n$$

With boundary conditions $a_0 = 0$ and $a_1 = 1$. Solve the relation finding a closed formula for a_n.

4. **Graphs:** The complete k-partite graph $K_{n_1,\ldots,n_k}$ has vartices partitioned into k subsets of size n_1, $n_2, \ldots, n_k$ repectively. Two vertices are adjacent if and only if they belong to different subsets of the partition.

   (a) (1 point) Draw a sketch for $K_{1,2,3}$.

   (b) (3 points) Give the number of vertices, edges and the degree of every vertex in $K_{100,100,100}$.

   (c) (2 points) Give necessary and sufficient conditions for K_{n_1,n_2,n_3} to have an Euler circuit.

   (d) (2 points) Show that $K_{1,2,3}$ has an Hamilton cycle.

5. **Minimal spanning trees:** (3 points) Find a minimal spanning tree for the graph in Figure 1 and give its total weight. In order to get full points you have to state which algorithm you are using (Kruskal or Prym) and show the iterations at every step.

6. **Transport Networks:** Consider the transport network in Figure 2, where S is the source and T is the sink.

   (a) (3 points) Find a flow with the maximum value for the network.

   (b) (2 points) Give a cut with the minimum capacity for the network. Determine the capacity of such cut.

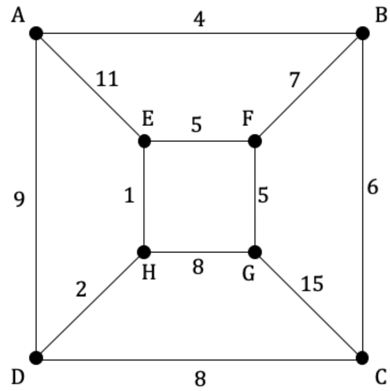

Figure 1: Weighted Graph

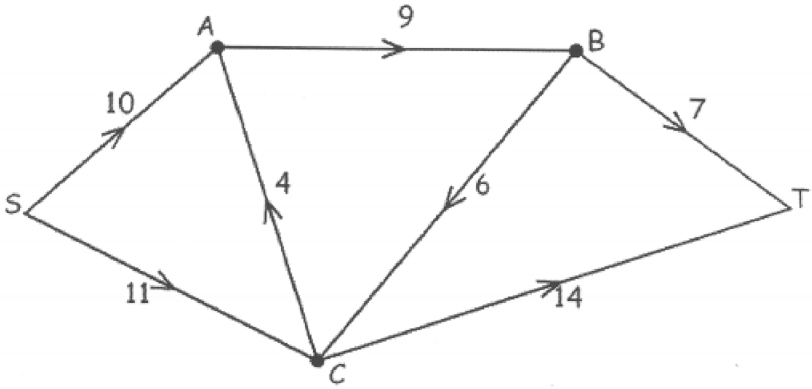

Figure 2: Network