

Solutions to Linear Algebra and Learning from Data October 26, 2023

1. The Hessian matrix is $H = \frac{1}{2}(A + A^\top)$. Using then the Jacobi method to check the positivity of the principal minors of H which shows $a \neq 2$. When $a > 2$, H is positive semi-definite so its smallest eigenvalue is 0.
2. (i) See Solutions to the exam 2022-10-25 1(i).
 (ii) See Solutions to the exam 2022-10-25 2(iv).
 (iii) Since $\|A\|_2$ is the largest singular value of A , which is the largest eigenvalue of A in absolute value if A is symmetric, the statement follows.
 (iv) $\|uu^\top - vv^\top\|_F^2 = \text{tr}((uu^\top - vv^\top)(uu^\top - vv^\top)^\top) = 2 - 2(u^\top v)^2 = 2 \sin^2 \alpha$
3. Note that

$$\begin{pmatrix} I & 0 \\ -C^\top S^{-1} & I \end{pmatrix} H \begin{pmatrix} I & -S^{-1}C \\ 0 & I \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & -C^\top S^{-1}C \end{pmatrix} =: \hat{H},$$

showing that H and \hat{H} have the same number of positive, negative and zero eigenvalues. Since $S > 0$ and C is invertible the matrix \hat{H} has exactly n positive and n negative eigenvalues and so does H , also proving H is indefinite.

4. (i) Clearly $AM = MC$. Then M is invertible we get the desired result.
 (ii) By inspection, we can choose $u = (-p_0 - 1, -p_1, \dots, -p_{n-1})^\top$, $v = e_n$.
 (iii) Note that $Q = C_{e_2}$ is an orthogonal matrix so

$$C = C_{e_2} + uv^\top = Q \underbrace{(I + Q^\top uv^\top)}_R,$$

where apparently $R := I + Q^\top uv^\top$ is an upper triangular matrix. Thus we have $Q_1 = Q, R_1 = R$. Notice that C has rank one update.

- (iv) Now $C_1 = R_1 Q_1 = Q + \underbrace{(Q^\top u)}_{u_1} \underbrace{(v^\top Q)}_{v_1^\top}$, which is again a rank one update. By more inspection we see that $v_1 = e_{n-1}$. Furthermore we rewrite $C_1 = Q(I + (Q^\top u_1)e_{n-1}^\top)$. Now the matrix $I + (Q^\top u_1)e_{n-1}^\top$ is an upper Hessenberg matrix we can perform a QR (for example via a Givens rotation G_1). This is a well-known method, the QR algorithm, for eigenvalue computation. Due to the special structure of the matrix C we would have a much cheaper QR procedure.
- (v) It is a straightforward check. Do it carefully.

- 5.(i-iii) This is a Toeplitz matrix but its inverse is not, see Lecture notes Day 11 at the course page.

(iv)

$$\|T_n^{-1}(\rho)\|_\infty = \frac{1 + 2|\rho| + \rho^2}{|1 - \rho^2|}$$

(v) By the Gershgorin circle theorem the eigenvalues must lie in the union of the discs centered at 1 with the radii

$$|\rho| + |\rho|^2 + \cdots + |\rho|^{n-1}, 2|\rho| + |\rho|^2 + \cdots + |\rho|^{n-2}, \dots, |\rho|^k + 2(|\rho| + \cdots + |\rho|^k),$$

where $k = \lfloor n/2 \rfloor$. Hence, an estimate for the spectral radius of $T_n(\rho)$ is the maximum of these numbers, which is bounded above by $|\rho| + |\rho|^2 + \cdots + |\rho|^{n-1} = \frac{|\rho|(1-|\rho|^{n-1})}{1-|\rho|}$, if $|\rho| < 1$ and by $|\rho|^k + 2(|\rho| + \cdots + |\rho|^k) = |\rho|^k + \frac{2|\rho|(1-|\rho|^{k-1})}{1-|\rho|}$, if $|\rho| > 1$. Thus an estimate for the spectral radius is $1 + \frac{|\rho|(1-|\rho|^{n-1})}{1-|\rho|}$, if $|\rho| < 1$ and by $1 + |\rho|^k + \frac{2|\rho|(1-|\rho|^{k-1})}{1-|\rho|}$, if $|\rho| > 1$. If $|\rho| = 1$ then the spectral radius is 1.