All solutions can be found in either the text book or the lecture notes.

MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
Linear Algebra and
Learning from Data AN, 7,5 hp
October 25, 2024

You are allowed to bring an A4 sheet (double sides) with whatever you think is important.
Don't forget to give proper arguments when you make statements!

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1. Let $A \in \mathbb{R}^{n\times n}$.

   (i) Show that $\|A\|_2^2 = \rho(A^\top A)$, where $\rho(A)$ is the spectral radius of A.
   (ii) Show that $\rho(A) = \|A\|_2$ if A is symmetric.
   (iii) If $p_m(x)$ is any real polynomial of degree m in x, and A is symmetric, show that $\|p_m(A)\|_2 = \rho(p_m(A))$.

   12 p

2. Let $H = \begin{pmatrix} S & A^\top \\ A & 0 \end{pmatrix}$ be a $2n \times 2n$ matrix with S being an $n \times n$ symmetric positive definite matrix. Is the function $f(x) = x^\top Hx$ convex where $x = (x_1, ..., x_{2n})$ are real variables? 12 p

3. Consider the equation $Ax = b$ with

$$A = \begin{pmatrix} 2 & 10 & 0 & -1 \\ 0 & -1 & 1 & 5 \\ 5 & 1 & 0 & 0 \\ -1 & 0 & 10 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 30 \\ 25 \\ 10 \\ 70 \end{pmatrix}.$$

   Provide an iterative method in the form $x^{(k+1)} = Bx^{(k)} + c$ that converges to the exact solution. Note that you must take this special problem into account. What is the convergence rate?

   12 p

4. Let $p(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_0$ and

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & 0 & \cdots & 0 & -p_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -p_{n-1} \end{pmatrix}.$$

   (i) Show that the polynomial $p(x)$ is both the characteristic and minimal polynomial for C.
   (ii) Show that C can be expressed as a unitary matrix plus a rank-1 matrix.
   (iii) Find all singular values of C.

   12 p

5. Let $A \in \mathbb{R}^n$ with component a_{ij} and let $D_i = \{z \in \mathbb{C} : |z - a_{ii}| \le r_i\}$. where $r_i = \sum_{j\neq i} |a_{ij}|$. We know that all eigenvalues of A lie in the union of D_i for $i = 1, ..., n$.

   (i) Assume A is diagonalizable and consider the perturbed matrix $A(\epsilon) = A + \epsilon B$ during a numerical computation ($\epsilon > 0$ is small). Show that

$$|\lambda_i(\epsilon) - \lambda_i| = O(\epsilon),$$

   where λ_i and $\lambda_i(\epsilon)$, $i = 1, ..., n$, are the eigenvalues of A and the eigenvalues of $A(\epsilon)$, respectively.
   (ii) Argue why eigenvalue problems are well-conditioned if the matrix is symmetric.
   (iii) Is the following statement correct? Eigenvalue problems can always be made well-conditioned if A is a tridiagonal matrix with a_i, $(i = 1, ..., n)$ as diagonal elements and b_i, c_i, $(i = 1, ..., n-1)$ upper and lower off-diagonal elements, respectively, where $b_i \neq c_i$ and $b_i c_i > 0$.

   12 p

The student office will inform you when the graded exam is ready for review.