MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
Linear Algebra and
Learning from Data AN, 7,5 hp
December 11, 2024

You are allowed to bring an A4 sheet (double sides) with whatever you think is important.
Don't forget to give proper arguments when you make statements!

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1. Let A be a symmetric matrix. We know that the following things are equivalent: (a) All eigenvalues of A are positive; (b) $x^\top Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$; (c) All leading principal minors of A are positive; (d) All pivots are positive.

   (i) Do these equivalent statements hold true if A has nonnegative eigenvalues? Give a complete statement and give an example in each case it fails to hold.

   (ii) Show that if the diagonal elements of a symmetric matrix A are zero, then all eigenvalues of A are nonnegative if and only if $A = 0$.

   (iii) Consider the sequence $\{c_0, c_1, ..., c_n\}$. Assume that not all c_k is zero. Show that if $\sum_{\alpha,\beta=0}^{n} c_{\alpha-\beta} x_\alpha x_\beta \geq 0$ then $c_0 > 0$.

   12 p

2. Assume that $\{p_i(x)\}_{i=0}^n$ are polynomials with p_i having degree i. Determine the determinant of the matrix $\begin{pmatrix} p_0(x_0) & p_1(x_0) & p_2(x_0) & \cdots & p_n(x_0) \\ p_0(x_1) & p_1(x_1) & p_2(x_1) & \cdots & p_n(x_1) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ p_0(x_n) & p_1(x_n) & p_2(x_n) & \cdots & p_n(x_n) \end{pmatrix}$.
   Hint: What is the answer if $p_i(x) = x^i$.

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3. Consider the least square problem: Minimize $\|b - Ax\|_2^2$, where $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$ are known and $x \in \mathbb{R}^n$ is unknown.

   (i) Show that the normal equation is necessary and sufficient condition for optimum.

   (ii) Derive the solution to the ℓ_2-regularized least square problem.

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4. (i) Let $A \in \mathbb{R}^{n\times n}$ be symmetric in the form $A = \begin{pmatrix} B & b \\ b^\top & c \end{pmatrix}$, where b is an $(n-1)$ column vector and c is a scalar. Assume that A has eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and B has eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1}$. Show that $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$.

   (ii) Let $C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & 0 & \cdots & 0 & -p_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -p_{n-1} \end{pmatrix}$. Find all eigenvalues of $C^\top C$.

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5. We know that every square matrix A is similar to an upper triangular matrix by a unitary transformation.

   (i) Use the above statement to prove the Caylay-Hamilton theorem.

   (ii) Find the relationship between the minimal polynomial and the characteristic polynomial of A. (Do not forget to prove your statement).

   (iii) Determine the first positive integer k such that $A^k = \sum_{j=0}^{k-1} \alpha_j A^j$ by applying Gram-Schmidt orthogonlization procedure with the standard inner product $\langle A, B\rangle = \text{trace}(A^*B)$. Write down the minimal polynomial of A.

   (iv) Determine $\alpha_0, ..., \alpha_{k-1}$ in the previous subproblem using QR factorization.

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The student office will inform you when the graded exam is ready for review.