MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
Linear Algebra and
Learning from Data AN, 7,5 hp
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You are allowed to bring an A4 sheet (double sides) with whatever you think is important.
Don't forget to give proper arguments when you make statements! If you have doubt on a problem formulation please write down your arguments.

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1. Assume that $A \in \mathbb{R}^{m\times n}$ has rank r.

   (i) Let the SVD of A be $A = U\Sigma V^\top$ with $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ being orthogonal. Determine the largest singular value of $Q = U_r V_r^\top$ where U_r is formed by the first r columns of U and V_r, the first r columns of V, respectively.

   (ii) Let now $m = n$. Show that $\rho(A) \le \|A\|$ for any matrix norm satisfying the property that $\|Ax\| \le \|A\|\|x\|$, where $\rho(A)$ is the spectral radius of A. When does the equality $\rho(A) = \|A\|$ hold?

   (iii) Consider the system of equations

$$\begin{aligned} 2x_1 + 10x_2 - x_3 - x_4 &= 30 \\ x_1 - x_2 + x_3 + 5x_4 &= 25 \\ 5x_1 + x_2 - 2x_3 - x_4 &= 10 \\ -x_1 + 2x_2 + 10x_3 - x_4 &= 70. \end{aligned}$$

   Provide a convergent iterative method to solve this system of equations. What is the convergence rate?

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2. (i) Show that any $A \in \mathbb{R}^{n\times n}$ can be split to $A = S + K$ were $S^\top = S$ and $K^\top = -K$. Find a pair of S and K explicitly. And show that

$$\|S + K\|_F^2 = \|S\|_F^2 + \|K\|_F^2.$$

   (ii) Consider the problem of minimizing $\|A - X\|$ over symmetric X for some suitable norm. Show that $\dfrac{A + A^\top}{2}$ is a solution. What norms are the suitable norms? Is it a unique solution? Provide a proof and/or a counterexample.

   (iii) In some cases, lack of symmetry is caused by rounding errors. The natural way to symmetrize A is to replace it by $\dfrac{A + A^\top}{2}$. Is this the best choice?

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3. We know that for any $A, B \in \mathscr{S}^n$

$$\lambda_{k+l-1}(A + B) \le \lambda_k(A) + \lambda_l(B),\ \forall k, l \in \{1, .., n\} \text{ such that } k + l - 1 \le n,$$

   where $\lambda_k(\cdot)$ stands for the k-th eigenvalue of the matrix "$\cdot$" in a decreasing order. (Take it as granted.)

(i) Show that

$$|\lambda_k(A) - \lambda_k(B)| \leq \|A - B\|_2, \ \forall k.$$

(ii) Argue why eigenvalue problems are well-conditioned if the matrix is symmetric.

(iii) Is the following statement correct? The eigenvalue problem is essentially well-conditioned if A is a tridiagonal matrix with a_i, $(i = 1, ..., n)$ as diagonal elements and b_i, c_i, $(i = 1, ..., n-1)$ upper and lower off-diagonal elements, respectively, where $b_i \neq c_i$ and $b_i c_i > 0$.

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4. Let $p(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_0$ and

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & 0 & \cdots & 0 & -p_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -p_{n-1} \end{pmatrix}.$$

(i) Show that the polynomial $p(x)$ is the characteristic polynomial for C.

(ii) Let an n by n matrix A and an n-vector b be such that the Krylov matrix

$$K = (b \ Ab \ A^2b \ \cdots \ A^{n-1}b)$$

is nonsingular. Show that A is similar to a companion matrix of the form C above with the last column only depending on A.

(iii) Show that C can be expressed as a sum of a unitary matrix and a rank-1 matrix. Does this observation have any numerical significance for eigenvalue computation?

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5. We learned that to study the quick decaying of singular values of a matrix we need pass certain test. The contents of the first two subproblems below belong to such a test for the Krylov matrix $K = (b \ Ab \ A^2b \ \cdots \ A^{n-1}b)$.

(i) Show that there is a matrix B such that $AK - KB$ is a rank-1 matrix (i.e. low rank). Hint: Choose B as the companion matrix in the previous problem with special choices of the last column.

(ii) Show that if n is even and A is Hermitian, then the spectrum of A and the spectrum of B are disjoint.

(iii) Show that the Vandermonde matrix $V = (x_j^i)$, $i = 0, ..., n-1$, $j = 1, ..., n$ with x_j real is a Krylov matrix with A Hermitian. Which A and b generate $K(= V)$?

(iv) It is shown that a real $n \times n$ Hankel matrix H_n is positive definite if and only if

$$(H_n)_{jk} = \sum_{\ell=1}^{n} \rho_\ell \theta_\ell^{j+k-2}, \ 1 \leq j, k \leq n$$

with $\rho_\ell > 0$ and θ_ℓ, $\ell = 1, ..., n$, are distinct real numbers. (Take it as granted) Show that there is a Krylov matrix K such that $H = K^*K$. Determine the K explicitly and show that $\sigma_j(H) = |\sigma_j(K)|^2$.

(v) What can you say about singular values of a Toeplitz matrix?

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The student office will inform you when the graded exam is ready for review.