MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Linear Algebra and
Learning from Data AN, 7,5 hp
December 9, 2025

You are allowed to bring an A4 sheet (double sides) with whatever you think is important.
Don't forget to give proper arguments when you make statements! If you have doubt on a problem formulation please write down your arguments.

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1. Assume that $A \in \mathbb{R}^{m\times n}$ has rank r.

   (i) Let the SVD of A be $A = U\Sigma V^\top$ with $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ being orthogonal. Determine $\|UV^\top\|_2$ and $\|UV^\top\|_F$.

   (ii) Let now $m = n$. Justify the statement that $\rho(A) \le \|A\|$ for all matrix norms.

   (iii) Justify the equality $\rho(A) = \|A\|$ for any matrix norms

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2. Consider the system of equations

$$\begin{aligned} 2x_1 + 10x_2 - x_3 - x_4 &= 30 \\ x_1 - x_2 + x_3 + 5x_4 &= 25 \\ 5x_1 + x_2 - 2x_3 - x_4 &= 10 \\ -x_1 + 2x_2 + 10x_3 - x_4 &= 70. \end{aligned}$$

   (i) Find an equivalent systems of equations $Ax = b$ for which we can split the matrix $A = A_1 + A_2$ so that the estimation of $\rho(B)$ can be done without computing the eigenvalues of B where $B = A_1^{-1}A_2$.

   (ii) Take your A. Estimate $\|A\|_2$ without computation of eigenavlues.

   (iii) Provide a convergent iterative method to solve this system of equations. What is the convergence rate?

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3. (i) State the Eckhart-Young theorem.

   (ii) Prove the Eckhart-Young theorem using the theorem that *for any* $A, B \in \mathscr{S}^n$

$$\lambda_{k+l-1}(A+B) \le \lambda_k(A) + \lambda_l(B),\ \forall k,l \in \{1,..,n\} \text{ such that } k+l-1 \le n,$$

   where $\lambda_k(\cdot)$ *stands for the* k*-th eigenvalue of the matrix "$\cdot$" in a decreasing order.*

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4. Let (A, b) be a pair of matrices $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ such that

$$K = (b\ Ab\ A^2b\ \cdots\ A^{n-1}b)$$

   has full rank.

(i) Show that matrix A is similar to C

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & 0 & \cdots & 0 & -p_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -p_{n-1} \end{pmatrix}.$$

and that the last column C only depends on A.

(ii) Show that the matrix $\begin{pmatrix} q \\ qA \\ \vdots \\ qA^{n-1} \end{pmatrix}$ is nonsingular where q is the last row of the matrix K^{-1}.

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5. We learned that to study the quick decaying of singular values of a matrix we need pass certain test. The contents of the first two subproblems below belong to such a test for the Krylov matrix $K = (b\ Ab\ A^2b\ \cdots\ A^{n-1}b)$.

(i) Show that there is a matrix B such that $AK - KB$ is a rank-1 matrix (i.e. low rank). Hint: Choose B as the companion matrix in the previous problem with special choices of the last column.

(ii) Show that if n is even and A is Hermitian, then the spectrum of A and the spectrum of B are disjoint.

(iii) Show that the Vandermonde matrix $V = (x_j^i)$, $i = 0, ..., n-1$, $j = 1, ..., n$ with x_j real is a Krylov matrix with A Hermitian. Which A and b generate $K(= V)$?

(iv) It is shown that a real $n \times n$ Hankel matrix H_n is positive definite if and only if

$$(H_n)_{jk} = \sum_{\ell=1}^{n} \rho_\ell \theta_\ell^{j+k-2},\ 1 \le j, k \le n$$

with $\rho_\ell > 0$ and θ_ℓ, $\ell = 1, ..., n$, are distinct real numbers. (Take it as granted) Show that there is a Krylov matrix K such that $H = K^*K$. Determine the K explicitly and show that $\sigma_j(H) = |\sigma_j(K)|^2$.

(v) What can you say about singular values of a Toeplitz matrix?

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The student office will inform you when the graded exam is ready for review.