MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Linear Algebra and
Learning from Data AN, 7,5 hp
October 28, 2026

You are allowed to bring an A4 sheet (double sides) with whatever you think is important.
Don't forget to give proper arguments when you make statements!

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1. Answer using False, or True, or Inconclusive, and justify your answer briefly.

   (i) Any matrix can be written as a sum of rank.one matrices.

   (ii) The matrix $(A^+A)^2 = A^+A$ is a projection.

   (iii) Singular values are invariant under basis changes.

   (iv) We used the following lemma to prove the Vandermonde factorizaton of a Hermitian positive semi-definite Toeplitz matrix.
   Lemma. If $X, Y \in \mathbb{C}^{m\times r}$ satisfy $XX^* = YY^*$, then $Y = XQ$ for some unitary $Q \in \mathbb{C}^{r\times r}$.
   Is the following proof correct?
   Write the SVDs $X = U_X\Sigma_X V_X^*$ and $Y = U_Y\Sigma_Y V_Y^*$. Then $XX^* = U_X\Sigma_X\Sigma_X^*U_X^*$ and $YY^* = U_Y\Sigma_Y\Sigma_Y^*U_Y^*$. Since these are equal, $U_X = U_Y$ and $\Sigma_X = \Sigma_Y$, so $Y = XV_XV_Y^*$.

10 p

2. Let $A = \begin{pmatrix} 1 & 2 & 4 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 3 & 6 & 3 \end{pmatrix}$.

   (i) Perform a rank full factorization $A = CR$, that is, C is of full column rank and R is of full row rank. Is this decomposition unique?

   (ii) Find the invertible matrix B such that $A = YBZ$ with the matrix B invertible without computing inverse. Is it unique? If not find such a factorization with B that has the largest determinant among all the same size of submatriices of A.

10 p

3. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x) = x^\top Ax$ where $A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 2 & \theta \end{pmatrix}$. What is the Hessian of f? For what values of θ is f strictly convex? 10 p

4. Let Z be the $n \times n$ matrix with ones on the first subdiagonal and zeros everywhere else:

$$Z_{ij} = \begin{cases} 1 & i = j + 1, \\ 0 & \text{otherwise}, \end{cases}$$

   Let $T = (t_{i-j})$, $i, j = 1, ..., n-1$, be Toeplitz matrix and $H = (h_{i+j})$, $i, j = 0, ..., n-1$.

   (i) What is the rank of $T - ZTZ^\top$?

(ii) What is the rank of $ZH - HZ^T$?

10 p

5. Let $q(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be monic, with companion matrix (bottom-row form)

$$C_q = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -c_0 & -c_1 & \cdots & -c_{n-1} \end{pmatrix},$$

and $a(x) = (1, x, \ldots, x^{n-1})^T$, $A = (a(x_1), \cdots, a(x_n))$.

(i) Show that that $C_q\, a(x) = x\, a(x) - q(x)\, e_2$. Then deduce $C_q = AXA^{-1}$ when q has distinct roots.

(ii) Compute all singular values of C_q for $n = 3$.

10 p

6. Let $A = \begin{pmatrix} 1-\beta-\alpha\lambda & -\beta \\ 1 & 0 \end{pmatrix}$, $0 < m \leq \lambda \leq L$, and $\alpha > 0, \beta > 0$. Determine the optimal spectral radius of A. 10 p

Good luck!

The student office will inform you when the graded exam is ready for review.