MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
Optimering AN, 7,5 hp
January 10, 2023

You are allowed to bring an A-4 sheet (double sides) with whatever you think is important.
You must motivate well your arguments.

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1. (i) Show that a set is convex if and only if its intersection with any line is convex.
   (ii) Consider the set $S = \{x : x_1^2 + x_2^2 \le 4\}$. Represent S as the intersection of a collection of half-spaces. Find the half-spaces explicitly. Is S a polyhedron? Find all extreme points of S.
   (iii) Let X be a nonempty bounded subset of $\mathbb{R}^n$. Define a function $S_X : \mathbb{R}^n \to \mathbb{R}$ by $S_X(x) = \sup\{\langle y, x\rangle : y \in X\}$. Show that $S_X(x)$ is a convex function.
   (iv) Assume that C and D are closed convex sets in $\mathbb{R}^n$. Show that $C = D$ if and only if $S_C(x) = S_D(x)$.
   (v) Determine $S_C(x)$ if $C \subset \mathbb{R}^n$ is a cone. 13 p

2. Find optimal solution to the following problem: Minimize $-x_1$ subject to $x_2 - (1 - x_1)^3 \le 0$ and $x_2 \ge 0$. Do the KKT conditions hold at the optimum?

   13 p

3. Derive a dual problem for

$$\text{Minimize} \sum_{i=1}^{N} \|A_i x + b_i\|_2 + \frac{1}{2}\|x - x_0\|_2^2.$$

   The problem data are $A_i \in \mathbb{R}^{m_i \times n}$, $b_i \in \mathbb{R}^{m_i}$, and $x_0 \in \mathbb{R}^n$.

   13 p

4. Consider the optimization problem to minimize $f(x)$ subject to $g_i(x) \le 0$ for $i = 1, ...m$.
   (i) Show that verifying whether a point $\bar{x}$ is a KKT point is equivalent to finding a vector u such that $A^t u = c, u \ge 0$ where $A \in \mathbb{R}^{m\times n}$.
   (ii) Use this to check whether $\bar{x} = (1, 2, 5)^t$ is a KKT point to the following problem:

$$\begin{aligned} \text{Minimize} \quad & 2x_1^2 + 2x_2^2 + 2x_3^2 + x_1x_3 - x_1x_2 + x_1 + 2x_3 \\ \text{subject to} \quad & x_1^2 + x_2^2 - x_3 \le 0 \\ & x_1 + x_2 + 2x_3 \le 16 \\ & x_1 + x_2 \ge 3 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

   13 p

5. Consider the piecewise-linear minimization problem: Minimize $f(x) = \max_{i=1,\dots,m}(a_i^t x + b_i)$ with variable $x \in \mathbb{R}^n$. Is it a linear program problem? Is this a convex program problem? Formulate this minimization problem as an LP problem. 12 p

You have finished the exam if your homework $p_h \ge 24$. Continue otherwise.

6. Consider the piecewise-linear minimization problem in the previous problem, call it (PWL). Suppose we approximate the objective function $f(x)$ by the smooth function

$$f_0(x) = \log\left(\sum_{i=1}^{m} \exp(a_i^t x + b_i)\right),$$

and solve the unconstrained problem (GP):

$$\text{minimize } \log\left(\sum_{i=1}^{m} \exp(a_i^t x + b_i)\right).$$

Find the dual problem as explicit as possible. Next let p^*_{pwl} and p^*_{gp} be the optimal values of (PWL) and (GP), respectively. Show that $0 \le p^*_{\text{gp}} - p^*_{\text{pwl}} \le \log m$. 12 p

You have finished the exam if your homework $23 \ge p_h \ge 16$. *Continue otherwise.*

7. (i) Let $G(x) = (\prod_{i=1}^n x_i)^{\frac{1}{n}}$ and $A(x) = \frac{1}{n}\sum_{i=1}^n x_i$ where $x_i > 0$ for $i = 1, ..., n$. Show, using the theory developed in this course, $G(x) \le A(x)$.
   (ii) Justify if the set $\{x \in \mathbb{R}^n_{++} : G(x) \ge A(x)\}$ is convex or not. Is this set a cone if we define $0^{\frac{1}{n}} = 0$?

12 p

You have finished the exam if your homework $15 \ge p_h \ge 8$. *Continue otherwise.*

8. (i) Using the Projection Theorem to find the solution of the quadratic programming problem:

$$\text{minimize } \left\{\frac{1}{2}\|x\|_2^2 + c^t x : Ax = 0\right\},$$

   where $A \in \mathbb{R}^{m\times n}$ with rank m and $c \in \mathbb{R}^n$ are given.
   (ii) Solve the following problem

$$\text{minimize } \left\{\frac{1}{2}(x-\bar{x})^t Q(x-\bar{x}) + c^t(x-\bar{x}) : Ax = b\right\},$$

   where A and c are as before, and $Q \in \mathbb{S}^n_{++}$ and $b \in \mathbb{R}^m$ are given.

12 p

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