Topology MM7052, HT22.

Solutions to Exam 2022-12-15

(1) (a) $\emptyset$ and $\mathbb{Z}$ are clearly open. If $\{U_i\}_{i\in I}$ is a family of open sets, then $\cup_i U_i$ is open because $n \in \cup_i U_i \Leftrightarrow n \in U_i$ for some $i \in I \Leftrightarrow -n \in U_i$ for some $i \in I \Leftrightarrow -n \in \cup_i U_i$. Similarly, $n \in \cap_i U_i \Leftrightarrow -n \in \cap_i U_i$ (regardless of whether or not I is finite).

(b) Not Hausdorff: if U, V are open sets such that $-1 \in U$ and $1 \in V$ then $1 \in U$ so $U \cap V \neq \emptyset$.
Not compact: $\mathbb{Z} = \bigcup_{n\in\mathbb{Z}}\{-n, n\}$ is an open cover that lacks a finite subcover.
Second countable: $\mathcal{B} = \{\{-n, n\} \mid n \in \mathbb{Z}\}$ is a countable basis for the topology because each $\{-n, n\}$ is open and every open set U can be written as $U = \bigcup_{n\in U}\{-n, n\}$.

(2) (a) $p|_A : A \to p(A)$ is clearly surjective so we need to check that

$$U \subseteq p(A) \text{ open} \Leftrightarrow (p|_A)^{-1}(U) \subseteq A \text{ open}.$$

$\Rightarrow$: this is just saying that $p|_A$ is continuous, which is clear as it is obtained from p by restricting the domain and codomain.
$\Leftarrow$: Since A is open in X by assumption, $(p|_A)^{-1}(U)$ is also open in X. Since p is an open map, it follows that $U = p((p|_A)^{-1}(U))$ is open in Y and hence also in $p(A)$.

(b) For example, let $X = \{a, b, c\}$ with open sets $\emptyset$, $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, let $Y = \{x, y\}$ with open sets $\emptyset$, $\{x, y\}$, define $p: X \to Y$ by $p(a) = p(c) = x$, $p(b) = y$, and let $A = \{a, b\}$. Then p is a quotient map but $p|_A : A \to p(A)$ is not.

(3) (a) If $A \subseteq \mathbb{R}$ contains two points $a, b \in A$ where $a < b$, then $(-\infty, a] \cap A$ and $(a, \infty) \cap A$ are two non-empty disjoint open subsets of A whose union is A, showing A is disconnected. (Note that $(-\infty, a]$ and (a, ∞) are open because they can be written as the union of all sets of the form $(x, a]$ and $(a, x]$, respectively, for $x \in \mathbb{R}$.)

(b) Totally disconnected: Let $A \subseteq C$ and suppose $f, g \in A$ with $f \neq g$. This means that $f(n) \neq g(n)$ for some $n \in \mathbb{Z}$, so $f(n) = 0$ and $g(n) = 1$ or $f(n) = 1$ and $g(n) = 0$. Either way, $ev_n^{-1}(0) \cap A$ and $ev_n^{-1}(1) \cap A$ are two non-empty disjoint open subsets of A whose union is A, so A is disconnected.
Not discrete: the topology has a countable basis,

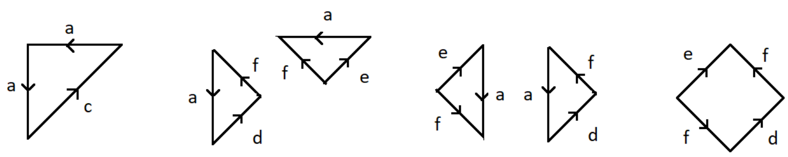
$$\left\{ev_{n_1}^{-1}(A_1) \cap \ldots \cap ev_{n_k}^{-1}(A_k) \mid A_i \subseteq \{0, 1\}, n_i \in \mathbb{Z}, k \geq 1\right\},$$

but the discrete topology does not as $\{0, 1\}^{\mathbb{Z}}$ is uncountable.

(4) We may present the torus as $T = I \times I/\sim$ where $(0, t) \sim (1, t)$ and $(t, 0) \sim (t, 1)$ for all $t \in I = [0, 1]$. The action of C_2 on T is induced by the action on $I \times I$ that flips the coordinates. Consider the triangle $\Delta = \{(x, y) \in I \times I \mid x \leq y\}$. The map $q: \Delta \to T/C_2$, defined as the composite of the inclusion $\Delta \to I \times I$ followed by the quotient maps $I \times I \to T \to T/C_2$, is a continuous surjective map from a compact space to a Hausdorff space (the orbit space T/C_2 is Hausdorff since C_2 is finite and T is Hausdorff), so it is a quotient map by the closed map lemma. Hence, it induces a homeomorphism

$$h: \Delta/\sim \to T/C_2,$$

where $p \sim p'$ if and only if $q(p) = q(p')$. The non-trivial identifications made by q are $(0, t) \sim (t, 1)$ for $t \in I$. This is exactly the polygonal presentation to the left in the figure below. The rest of the figure indicates a sequence of elementary transformations that transforms this to a standard presentation for the Möbius band.



(5) By the lifting criterion, a map $f\colon \mathbb{R}\mathrm{P}^n \to S^1$ lifts to the universal cover

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow p \\ \mathbb{R}\mathrm{P}^n & \xrightarrow{\ f\ } & S^1 \end{array}$$

if and only if $f_*(\pi_1(\mathbb{R}\mathrm{P}^n))$ is the trivial subgroup of $\pi_1(S^1) \cong \mathbb{Z}$. For $n \geq 2$ we have that $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}/2\mathbb{Z}$. This can be seen by identifying $\mathbb{R}\mathrm{P}^n$ with the orbit space associated to the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^n, using that the latter is a covering space action and that S^n is simply connected for $n \geq 2$. Since there are no non-zero homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$, this implies that $f_*(\pi_1(\mathbb{R}\mathrm{P}^n))$ must be the trivial subgroup of $\pi_1(S^1)$, so a lift $\widetilde{f}$ exists in this case. Since $\mathbb{R}$ is contractible, $\widetilde{f}$ is homotopic to a constant map. This implies that $f = p \circ \widetilde{f}$ is homotopic to a constant map.

(6) Let X denote the complement of the three coordinate axes in $\mathbb{R}^3$. The inclusion $i\colon S^2 \cap X \to X$ is a homotopy equivalence. Indeed, if we define $r\colon X \to S^2 \cap X$ by $r(x) = x/|x|$ and $H\colon X \times [0, 1] \to X$ by the formula $H(x, t) = (1-t)x + tx/|x|$, then $ri = 1$ and H is a homotopy from 1_X to ir. Next, observe that $S^2 \cap X$ is S^2 with six points removed (namely $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$). By stereographic projection from $(0, 0, 1)$ to the xy-plane, $S^2 \cap X$ is homeomorphic to $\mathbb{R}^2$ with five points removed (namely $(\pm 1, 0)$, $(0, \pm 1)$ and $(0, 0)$). The fundamental group of $\mathbb{R}^2 \setminus \{p_1, \ldots, p_r\}$, where $p_1, \ldots, p_r$ are distinct points in $\mathbb{R}^2$, is isomorphic to the r-fold free product $\mathbb{Z}^{*r}$, i.e., the free group on r generators. This can be shown by induction on r. The base case $r = 1$ follows from $\mathbb{R}^2 \setminus \{p_1\} \cong \mathbb{R}^2 \setminus \{0\} \simeq S^1$ and the fact that $\pi_1(S^1) \cong \mathbb{Z}$. Induction step: Let $r > 1$. If the points $p_1, \ldots, p_r$ are not on a vertical line, then we can find real numbers $a < b$ such that both $U' = \{(x, y) \in \mathbb{R}^2 \mid x > a\}$ and $V' = \{(x, y) \in \mathbb{R}^2 \mid x < b\}$ contain at least one of the points p_i but $U' \cap V'$ contains none of the points. Setting $U = U' \cap (\mathbb{R}^2 \setminus \{p_1, \ldots, p_r\})$ and $V = V' \cap (\mathbb{R}^2 \setminus \{p_1, \ldots, p_r\})$, we get that $U \cup V = \mathbb{R}^2 \setminus \{p_1 \ldots, p_r\}$, $U \cong \mathbb{R}^2 \setminus \{s \text{ points}\}$, $V \cong \mathbb{R}^2 \setminus \{t \text{ points}\}$, where $s + t = r$ and $s, t < r$, and $U \cap V = (a, b) \times \mathbb{R}$ is contractible. By the Seifert–van Kampen theorem and induction,

$$\pi_1(U \cup V) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \mathbb{Z}^{*s} *_{\{1\}} \mathbb{Z}^{*t} \cong \mathbb{Z}^{*r}.$$

If the points happen to be on a vertical line, then they are not on a horizontal line (since we assume $r > 1$) and one can argue as above using sets of the form $U' = \{(x, y) \in \mathbb{R}^2 \mid y > a\}$ and $V' = \{(x, y) \in \mathbb{R}^2 \mid y < b\}$.