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Avd. Matematik
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Tentamensskrivning i
Topologi MM7052
7,5 hp
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(1) (a) [1 pt] Show that the collection $\mathcal{B} = \{(a, \infty) \subset \mathbb{R} : a \in \mathbb{R}\}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}$.
(b) [1 pt] What are the closed sets in $\mathcal{T}$?
(c) [1 pt] What are the limit points of the set $(0, 1)$ with respect to $\mathcal{T}$?
(d) [1 pt] What are the path connected components of $\mathbb{R}$ with respect to $\mathcal{T}$?
(e) [1 pt] Determine if any of the sets $\{\frac{1}{n} : n \in \mathbb{N}\}$ and $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ are compact with respect to $\mathcal{T}$.

Solution: (a) First note that for all $x \in \mathbb{R}$ holds $x \in (x-1, \infty)$, so that $\mathcal{B}$ covers $\mathbb{R}$. Moreover, for any $a, b \in \mathbb{R}$, we have $(a, \infty) \cap (b, \infty) = (\max(a, b), \infty) \in \mathcal{B}$. Hence, by Proposition 2.44, the set $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

(b) We first claim that $\mathcal{T} = \mathcal{B} \cup \{\mathbb{R}, \varnothing\}$. To prove this claim, note that the open sets of $\mathcal{T}$ are precisely those that can be written as unions of elements of $\mathcal{B}$. In particular the inclusion "$\supseteq$" is clear. To prove the other inclusion, let $\mathcal{A} \subseteq \mathcal{B}$ be any subset and $A \subseteq \mathbb{R}$ the subset such that $\mathcal{A} = \{(a, \infty) \mid a \in A\}$. Write $U = \bigcup_{B \in \mathcal{A}} B$. If $\mathcal{A}$ is empty, then $U = \varnothing$. Otherwise, set $m = \inf_{a \in A} a$. If $m = -\infty$, i.e. if A is unbounded below, then $U = \mathbb{R}$. Otherwise $m \in \mathbb{R}$ and $U = (m, \infty)$ by definition of the infimum. This proves the claim.
The claim implies that the set of closed sets in $\mathcal{T}$ is precisely equal to $\{(-\infty, a] \mid a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$.

(c) We claim that the set of limit points of $(0, 1)$ in $\mathcal{T}$ is equal to $(-\infty, 1]$. Let $l \leq 1$. Then an open set U that contains l is either of the form (a, ∞) with $a < l \leq 1$, or $U = \mathbb{R}$. In either case $U \cap (0, 1) \supseteq (\max(0, a), 1) \neq \varnothing$, and hence l is a limit point of $(0, 1)$. On the other hand, let $k > 1$. Then $(\frac{k+1}{2}, \infty)$ is an open set containing k that does not intersect $(0, 1)$. Hence k is not a limit point of $(0, 1)$.

(d) We claim that $\mathbb{R}$ is path-connected with respect to $\mathcal{T}$, i.e. that the whole set $\mathbb{R}$ is the only path component. To prove this, it is enough to show that for all $a, b \in \mathbb{R}$ such that $a < b$ the map $\gamma \colon [0, 1] \to \mathbb{R}$ given by $\gamma(t) = (1-t)a + tb$ is continuous with respect to $\mathcal{T}$. Let $c \in \mathbb{R}$; then we have that

$$\gamma^{-1}((c, \infty)) = \begin{cases} [0, 1], & \text{if } c < a \\ (\frac{c-a}{b-a}, 1], & \text{if } a \leq c \leq b \\ \varnothing, & \text{if } b \leq c \end{cases}$$

is open in $[0, 1]$. Similarly $\gamma^{-1}(\mathbb{R}) = [0, 1]$ and $\gamma^{-1}(\varnothing) = \varnothing$ are open in $[0, 1]$. Hence γ is continuous.

(e) The set $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is not compact with respect to $\mathcal{T}$. To see this, note that the set $\mathcal{C} = \{(\frac{1}{n}, \infty) \mid n \in \mathbb{N}\}$ covers A since $\frac{1}{n} \in (\frac{1}{n+1}, \infty)$, but that no finite subset of $\mathcal{C}$ covers A.
The set $B = A \cup \{0\}$ is compact with respect to $\mathcal{T}$. To see this, let $\mathcal{D} \subseteq \mathcal{T}$ be a subset that covers B. Then there exists a $D \in \mathcal{D}$ such that $0 \in D$. Hence either $D = (d, \infty)$ with $d < 0$, or $D = \mathbb{R}$. In either case the finite subset $\{D\} \subseteq \mathcal{D}$ covers B.

(2) Let X be a topological space and define $\Delta : X \to X \times X$ by $\Delta(x) = (x, x)$.
(a) [2 pts] Show that Δ is continuous and then that it is an embedding.

(b) [3 pts] Show that $\Delta(X)$ is closed if and only if X is Hausdorff. (Note that this is a standard result, but you need to prove it and cannot just refer to a text book.)

Solution: (a) The map Δ is continuous by the characteristic property of the product, since both component functions are equal to the identity of X, which is continuous. Furthermore define $p = \mathrm{pr}_1|_{\Delta(X)} \colon \Delta(X) \to X$, i.e. the projection to the first coordinate, restricted to the image of Δ. The map p is continuous since the projection is continuous. Furthermore note that $p \circ \Delta$ is the identity of X and that $\Delta \circ p$ is the identity of $\Delta(X)$. Thus Δ is a homeomorphism onto its image, i.e. an embedding.

(b) That $\Delta(X) \subseteq X \times X$ is closed is equivalent to the existence, for all $(x, y) \in (X \times X) \setminus \Delta(X)$, of an element of the basis of $X \times X$ that contains (x, y) and does not intersect $\Delta(X)$. Note that $(x, y) \in (X \times X) \setminus \Delta(X)$ is equivalent to $x \neq y$. Furthermore, an element of the basis of $X \times X$ is of the form $U \times V$ with U and V open subsets of X, and it contains (x, y) if and only if $x \in U$ and $y \in V$. Lastly note that $(U \times V) \cap \Delta(X) = \varnothing$ is equivalent to $U \cap V = \varnothing$. Hence $\Delta(X)$ being closed is equivalent to the existence, for all $x, y \in X$ with $x \neq y$, of two open subsets U and V of X such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$. But this is the definition of X being Hausdorff.

(3) (a) [3 pt] Consider the cylinder $\mathbb{S}^n \times I$ and define $f : \mathbb{S}^n \times I \to \overline{\mathbb{B}}^{n+1}$ by $(x, t) \mapsto tx$. Show that f is a quotient map.

(b) [2 pt] Show that the quotient map $\mathbb{R}^{n+1} \setminus \{0\} \to (\mathbb{R}^{n+1} \setminus \{0\})/\sim$, where $x \sim y$ if $x = \lambda y$ for some non-zero $\lambda \in \mathbb{R}$, is not a covering map.

Solution: (a) The map f is a restriction of the map $g\colon \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ given by $g(x, t) = tx$. Since g is continuous, so is f. Furthermore f is surjective since we have, for any $b \in \overline{\mathbb{B}}^{n+1} \setminus \{0\}$, that $f(\frac{b}{|b|}, |b|) = b$ and that $f(s, 0) = 0$, where s is any element of $\mathbb{S}^n$. Moreover note that $\mathbb{S}^n \times I$ is compact, since both $\mathbb{S}^n$ and I are compact, and that $\overline{\mathbb{B}}^{n+1}$ is Hausdorff since it is a subspace of $\mathbb{R}^{n+1}$. Hence f is a quotient map by the closed map lemma.

(b) Consider the equivalence class x of the point $(1, 0, \dots, 0)$ in the quotient. The preimage of x under the quotient map is equal to the subspace $(\mathbb{R} \setminus \{0\}) \times \{0\}^n \subseteq \mathbb{R}^{n+1} \setminus \{0\}$. This subspace is not discrete and hence the quotient map is not a covering.

(4) [5 pts] Let G be a connected topological group with neutral element e and $f : \mathbb{R} \to G$ a continuous group homomorphism. For every $n \in \mathbb{Z}$ let $\alpha_n : I \to G$ be the path $\alpha_n(t) = f(nt)$. Prove that if $\mathbb{Z} \subset f^{-1}(e)$, the map $\mathbb{Z} \to \pi_1(G, e)$ defined by $n \mapsto [\alpha_n]$, is a group homomorphism.

Solution: Let $\mathbb{R}/\mathbb{Z}$ be the quotient group, and write $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ for the projection map. Since $\mathbb{Z} \subseteq \ker(f)$, there is a unique group homomorphism $g\colon \mathbb{R}/\mathbb{Z} \to G$ such that $g \circ \pi = f$. Equipping $\mathbb{R}/\mathbb{Z}$ with the quotient topology induced by π, we obtain that g is continuous by the characteristic property of the quotient topology. Moreover, let $\beta_n\colon I \to \mathbb{R}/\mathbb{Z}$ be the map given by $\beta_n(t) = \pi(nt)$, which is continuous as a composite of two continuous maps. Note that $\alpha_n = g \circ \beta_n$.

By Example 3.92, there is a homeomorphism $\phi\colon \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ given by $\phi([x]) = e^{2\pi i x}$. This induces a group isomorphism $\phi_*\colon \pi_1(\mathbb{R}/\mathbb{Z}, [0]) \cong \pi_1(\mathbb{S}^1, 1)$. Composing this with the group isomorphism $\mathbb{Z} \to \pi_1(\mathbb{S}^1, 1)$ given by $n \mapsto [\gamma_n]$ where $\gamma_n(t) = e^{2\pi i n t}$, we obtain a group isomorphism $\mathbb{Z} \to \pi_1(\mathbb{R}/\mathbb{Z}, [0])$ given by $n \mapsto [\beta_n]$ (since $\phi \circ \beta_n = \gamma_n$). Further composing this with g_*, we obtain a group homomorphism $\mathbb{Z} \to \pi_1(G, e)$ given by $n \mapsto [g \circ \beta_n] = [\gamma_n]$.

(5) [5 pts] Compute the fundamental group of the complement of n points on $\mathbb{S}^2$.

Solution: By the stereographic projection, the complement of a single point in $\mathbb{S}^2$ is homeomorphic to $\mathbb{R}^2$. Hence the complement of $n \geq 1$ points in $\mathbb{S}^2$ is homeomorphic to the complement of $n-1$ points in $\mathbb{R}^2$.

We now claim that the complement of $k \geq 0$ points in $\mathbb{R}^2$ is path-connected and that its fundamental group is isomorphic to the free product of k copies of $\mathbb{Z}$. We proceed by induction on k. The claim is true for $k = 0$ since $\mathbb{R}^2$ is contractible; in particular its fundamental group is trivial. Now assume that $k \geq 1$ and let A be the set of k points whose complement we are considering.

Let C be the convex hull of the finite subset $A \subseteq \mathbb{R}^2$; this is a non-empty convex polygon. Let a be a corner of this polygon (in particular $a \in A$). By rotating and scaling $\mathbb{R}^2$, we can assume that $a = (0,0)$ and that a is the only point of C (and hence the only point of A) whose first coordinate is ≤ 0. Then there exist $x, y \in \mathbb{R}$ such that $0 < x < y$ and $A \setminus \{a\} \subset (y, \infty) \times \mathbb{R}$.

Let $U = ((-\infty, y) \times \mathbb{R}) \setminus \{a\}$ and $V = ((x, \infty) \times \mathbb{R}) \setminus (A \setminus \{a\})$. Then U and V together are an open cover of $\mathbb{R}^2 \setminus A$. Moreover $U \cong \mathbb{R}^2 \setminus \{(0,0)\}$, $U \cap V = (x, y) \times \mathbb{R} \cong \mathbb{R}^2$, and V is homeomorphic to a complement of $k-1$ points in $\mathbb{R}^2$. Note that $U \simeq \mathbb{S}^1$; in particular U is path-connected and $\pi_1(U)$ is isomorphic to $\mathbb{Z}$. By induction V is path-connected and $\pi_1(V)$ is isomorphic to the free product of $k-1$ copies of $\mathbb{Z}$. Since $U \cap V$ is non-empty, this implies that $\mathbb{R}^2 \setminus A$ is path-connected. Moreover, since $U \cap V$ is contractible, we can apply the special case of the Seifert–van Kampen theorem given in Corollary 10.4 to deduce that $\pi_1(\mathbb{R}^2 \setminus A) \cong \pi_1(U) * \pi_1(V) \cong \mathbb{Z} * \pi_1(V)$, which is the free product of k copies of $\mathbb{Z}$.

(6) [5 pts] Prove the following theorems:

Theorem 1. *(Homotopy Classification of Loops in $\mathbb{S}^1$) Two loops in $\mathbb{S}^1$ based at the same point are path-homotopic if and only if they have the same winding number.*

Theorem 2. *(Fundamental Group of the Circle) The group $\pi_1(\mathbb{S}^1, 1)$ is an infinite cyclic group generated by the loop $\omega : I \to \mathbb{S}^1$ defined by $\omega(s) = e^{2\pi i s}$.*

Solution: These are Theorems 8.8 and 8.9 from the book, and their proofs can be found there.