MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonas Bergström

Tentamensskrivning i
Topologi MM7052
7,5 hp
2024-12-18

(1) Put $X = \{n \in \mathbb{N} : n \geq 2\}$ and, for each $m \in X$, put $S_m = \{n \in X : n \text{ divides } m\}$.
  (a) [1 pt] Show that the sets S_m, for $m \in X$, form a basis of a topology on X.
  (b) [1 pt] For any $n \in X$, compute the closure of the set $\{n\}$ in X.
  (c) [1 pt] Show that the subspace topology on $P \subset X$, where P consists of all prime numbers, is discrete.
  (d) [1 pt] Show that a subset of X is compact if and only if it is finite.
  (e) [2 pts] Show that X is path connected and locally path connected.
  Hint: Find paths between points $n \in X$ and points $nk \in X$ for $k \in \mathbb{N}$.

(2) Let X be a topological space and consider the following three statements:
  (1) X is Hausdorff.
  (2) If a sequence $(x_n)_{n=1}^{\infty}$ in X converges to a limit $x \in X$, then the limit is unique.
  (3) The set $\{x\}$ is closed in X for every $x \in X$.

  Show that:
  (a) [1 pt] (1) implies (2)
  (b) [1 pt] (2) implies (3)
  (c) [2 pts] (3) does not imply (2)
  (d) [2 pts] (2) does not imply (1).
  Hint: Consider for instance the topology on a set X for which $U \subseteq X$ is open iff $U = \emptyset$ or $X \setminus U$ is a countable set.

(3) (a) [3 pts] Let X be the union of $\mathbb{S}^2 \subset \mathbb{R}^3$ with $Y_1 = \{(x, 0, 0) : x \in [-1, 1]\}$ and $Y_2 = \{(0, y, 0) : y \in [-1, 1]\}$. Compute the fundamental group of X.
  (b) [3 pts] Let Z be the union of $\mathbb{S}^2 \subset \mathbb{R}^3$ with the three coordinate planes. Compute the fundamental group of Z.
  For this exercise you are free to argue a bit more intuitively (e.g. using pictures instead of formulas to define homotopies).

(4) Say that $f : X \to Y$ is a proper local homeomorphism, with X and Y locally compact and connected, Hausdorff spaces. Say also that X is non-empty and locally path connected.
  (a) [2 pts] Show that f is an open and closed map, and therefore surjective.
  (b) [1 pt] For each point $y \in Y$, show that $f^{-1}(y)$ consists of a finite set of points.
  (c) [2 pts] For each point $y \in Y$, show that there are disjoint open neighborhoods U_x of each point of $f^{-1}(y)$ mapping homeomorphically onto their image.
  (d) [1 pt] Show that $V = \bigcap_{x \in f^{-1}(y)} f(U_x) \setminus f(X \setminus \bigcup_{x \in f^{-1}(y)} U_x)$ is evenly covered, and use this to conclude that f is a covering map.

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(5) Prove the theorem of *Existence of the Universal Covering Space*. More precisely, fix a point x_0 in a locally simply connected topological space X, let $\tilde{X}$ be the set of path classes starting at x_0 and define a map $q : \tilde{X} \to X$ by $q([f]) = f(1)$. Then:

(a) [2 pts] Define a topology on $\tilde{X}$.

Assume it now to be known that $\tilde{X}$ is path connected.

(b) [2 pts] Show that q is a covering map.

(c) [2 pts] Show that $\tilde{X}$ is simply connected.