MATEMATISKA INSTITUTIONEN
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Avd. Matematik
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Tentamensskrivning i
Topologi MM7052
7,5 hp
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(1) Put $X = \{n \in \mathbb{N} : n \geq 2\}$ and, for each $m \in X$, put $S_m = \{n \in X : n \text{ divides } m\}$.
   (a) [1 pt] Show that the sets S_m, for $m \in X$, form a basis of a topology on X.
   (b) [1 pt] For any $n \in X$, compute the closure of the set $\{n\}$ in X.
   (c) [1 pt] Show that the subspace topology on $P \subset X$, where P consists of all prime numbers, is discrete.
   (d) [1 pt] Show that a subset of X is compact if and only if it is finite.
   (e) [2 pts] Show that X is path connected and locally path connected.
   Hint: Find paths between points $n \in X$ and points $nk \in X$ for $k \in \mathbb{N}$.

Solution:

(a) In $\mathbb{N}$, n divides both m and m' if and only if n divides $d = \gcd(m, m')$. Thus $S_m \cap S_{m'} = S_d$ if $d > 1$ and $\varnothing$ if $d = 1$. Each $m \in X$ belongs to S_m, so $\bigcup_{m \in X} S_m = X$. Therefore the collection $\{S_m \mid m \in X\}$ is a basis for a topology on X.

(b) It will be more convenient to first compute the interior of $\{n\}^c$, which is the union of all open sets contained in $\{n\}^c$. These open sets are exactly the unions of those basic open sets S_m that are contained in $\{n\}^c$, ie those S_m that do not contain n. In other words,

$$\text{Int}(\{n\}^c) = \bigcup_{n \text{ does not divide } m \in X} S_m.$$

But this union is exactly equal to $\{d \in X \mid n \text{ does not divide } d\}$: if n does not divide d then we can take $m = d$, and conversely if n divides d and $d \in S_m$ then n divides m so d cannot be in the above union. Thus

$$\overline{\{n\}} = (\text{Int}(\{n\}^c))^c = \{d \in X \mid n \text{ divides } d\}.$$

(c) Since distinct primes cannot divide each other, $S_p \cap P = \{p\}$ for each $p \in P$. Since S_p is open in X by definition, it follows that $\{p\}$ is open in the subspace topology on P, which implies that this topology is discrete.

(d) Any topology on a finite set F makes it compact since choosing an element of the open cover containing each point of F produces a finite subcover.
Conversely, suppose $K \subset X$ is compact and consider the cover of K by $\{S_m \mid m \in K\}$; this is a cover since $m \in S_m$ for $m \in X$. So K is covered by finitely many S_m, and since each S_m is finite, K is a subset of a finite union of finite sets, in particular is itself finite.

(e) Suppose $k, n \in X$ are such that k divides n. Now any open set containing n must contain some S_m such that n divides m and therefore must also contain k. It follows that the function $f : [0,1] \to \{k, n\} \subset X$ given by

$$f(t) = \begin{cases} k & \text{if } t < 1, \\ n & \text{if } t = 1. \end{cases}$$

is continuous, ie there is a path in $\{k, n\}$ from k to n: we have checked that $\{n\}$ is not open in $\{k, n\}$ and the preimage of each of the other three subsets of $\{k, n\}$ under f is open in $[0,1]$ by construction.
Now each S_m is path connected: if $n \in S_m$ then there is a path in $\{n, m\} \subset S_m$ connecting n and m. It follows that X is locally path connected: any open set containing $n \in \mathbb{N}$ contains some basic neighborhood S_m of n.

To see that all of X is path connected, consider any $m, n \in X$. By above, there are paths between m and mn and between mn and n. Concatenating, we get a path between m and n.

(2) Let X be a topological space and consider the following three statements:
   (1) X is Hausdorff.
   (2) If a sequence $(x_n)_{n=1}^{\infty}$ in X converges to a limit $x \in X$, then the limit is unique.
   (3) The set $\{x\}$ is closed in X for every $x \in X$.

   Show that:
   (a) [1 pt] (1) implies (2)
   (b) [1 pt] (2) implies (3)
   (c) [2 pts] (3) does not imply (2)
   (d) [2 pts] (2) does not imply (1).
   Hint: Consider for instance the topology on a set X for which $U \subseteq X$ is open iff $U = \emptyset$ or $X \setminus U$ is a countable set.

Solution:

(a) If $x, y \in X$ are distinct then they have disjoint neighborhoods U and V respectively. A sequence in X cannot eventually be in both U and V and therefore cannot converge to both x and y. Therefore limits of sequences in X are unique, if they exist.

(b) Let $x, y \in X$ be distinct. We want to show that there is a neighborhood of y that does not contain x. Consider the constant sequence $x_n = x$ in X. Every neighborhood of x contains every x_n so it converges to x and, since limits are unique, does not converge to y. So there must be a neighborhood U of y such that x_n is not eventually in U, but the only way this can happen is if $x \notin U$.

(c) Consider the cofinite topology on an infinite set X, ie a subset U is open if and only if $X \setminus U$ is finite or $U = \varnothing$. Since $\{x\}$ is finite, $X \setminus \{x\}$ is open, ie $\{x\}$ is closed. Consider a sequence (x_n) in X that takes any value at most once, ie an injection $\mathbb{N} \to X$ (which exists since X is infinite). Now for any $x \in X$ and any open set $U \ni x$, since $X \setminus U$ is finite there are only finitely many n such that x_n is in $X \setminus U$. In other words, x_n is in U for all but finitely many n. Since x and U are arbitrary, it follows that (x_n) converges to *every* point of X, in particular does not have a unique limit.

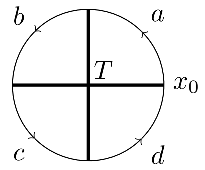
(d) We will use *countable* so as to include finite. Consider the cocountable topology on an uncountable set X, ie a subset U is open if and only if $X \setminus U$ is countable or $U = \varnothing$. Since the union of two countable sets is countable and therefore not all of X, the intersection of any two non-empty open sets of X is non-empty and so X cannot be Hausdorff. Consider any sequence (x_n) in X and let $A = \{x_n\}$, the collection of all the values taken by (x_n), which must be a countable set. Now for any $x \in X$, let $U = X \setminus (A \setminus \{x\})$, which is an open neighborhood of $\{x\}$. The only value in U that x_n can take is x so if x_n converges to x then we must have $x_n = x$ for large n. Thus there can be at most one such x.

(3) (a) [3 pts] Let X be the union of $\mathbb{S}^2 \subset \mathbb{R}^3$ with $Y_1 = \{(x, 0, 0) : x \in [-1, 1]\}$ and $Y_2 = \{(0, y, 0) : y \in [-1, 1]\}$. Compute the fundamental group of X.
   (b) [3 pts] Let Z be the union of $\mathbb{S}^2 \subset \mathbb{R}^3$ with the three coordinate planes. Compute the fundamental group of Z.
   For this exercise you are free to argue a bit more intuitively (e.g. using pictures instead of formulas to define homotopies).

Solution:

(a) Let Γ be the subspace $X \cap (\mathbb{R}^2 \times 0)$, which has fundamental group freely generated by the four edges a, b, c, d considered as loops at $x_0 = (1, 0, 0)$ via paths in the *tree* T

as in the following figure:



Further, X is obtained by attaching two disks to Γ along the boundary circle, which represents the element $abcd$ in $\pi_1(\Gamma, x_0) \cong \langle a, b, c, d \mid \varnothing \rangle$: since either hemisphere of the sphere is a disk, there is a continuous bijection $\Gamma \cup_{abcd} D^2 \cup_{abcd} D^2 \to X$ which must be a homeomorphism since the domain is compact and the target X is Hausdorff. Therefore

$$\pi_1(X, x_0) \cong \langle a, b, c, d \mid abcd \rangle \cong \langle a, b, c \mid \varnothing \rangle$$

where the second isomorphism is given by mapping $d \mapsto (abc)^{-1}$ (and a, b, c by "identity"). In words, the required fundamental group (observe that X is path connnected) is a free group on three generators.

(b) Similar to above, Z is obtained by attaching eight disks (one for each orthant) to the union of the coordinate planes, which we will denote by Z'. To see this, note that $Z \cap \overline{B}(0,2)$ (here $\overline{B}$ denotes the closed ball) is obtained by gluing eight disks to $Z' \cap \overline{B}(0,2)$ by the same argument as part (a): there is a continuous bijection from $Z' \cap \overline{B}(0,2) \cup \bigcup_8 D^2$, which is compact, to $Z \cap \overline{B}(0,2)$, which is Hausdorff. Further, this homeomorphism can be chosen to be identity on $Z' \cap \overline{B}(0,2)$ and therefore extends by identity to all of Z. But Z' is contractible by the map $(x,t) \mapsto tx$. Therefore $\pi_1(Z)$, being a quotient of $\pi_1(Z')$, is a trivial group.

(4) Say that $f : X \to Y$ is a proper local homeomorphism, with X and Y locally compact and connected, Hausdorff spaces. Say also that X is non-empty and locally path connected.
   (a) [2 pts] Show that f is an open and closed map, and therefore surjective.
   (b) [1 pt] For each point $y \in Y$, show that $f^{-1}(y)$ consists of a finite set of points.
   (c) [2 pts] For each point $y \in Y$, show that there are disjoint open neighborhoods U_x of each point of $f^{-1}(y)$ mapping homeomorphically onto their image.
   (d) [1 pt] Show that $V = \bigcap_{x \in f^{-1}(y)} f(U_x) \setminus f(X \setminus \bigcup_{x \in f^{-1}(y)} U_x)$ is evenly covered, and use this to conclude that f is a covering map.

Solution:

(a) Since f is a local homeomorphism it is an open map: any open set U of X can be written as $\bigcup U_\alpha$ such that $f : U_\alpha \to V_\alpha = f(U_\alpha)$ is a homoemorphism with open sets $V_\alpha \subset Y$, so $f(U) = \bigcup V_\alpha$ is open. On the other hand it is also a closed map: suppose $C \subset X$ is closed and $y \in Y \setminus f(C)$. Since Y is locally compact, there is a neighborhood V of y contained in a compact subset K of Y. Then $f^{-1}(K) \cap C$ is a closed subset of the compact set $f^{-1}(K)$ and therefore it is compact. It follows that $f(f^{-1}(K) \cap C)$ is also compact, hence closed (since Y is Hausdorff). Therefore $V \setminus f(f^{-1}(K) \cap C)$ is an open neighborhood of y. It suffices to check that $V \setminus f(f^{-1}(K) \cap C)$ is disjoint from $f(C)$. Pick an arbitrary $y' \in V \cap f(C)$, so $y' = f(x)$ for some $x \in C$. Since $f(x) = y' \in V \subset K$, in fact $x \in f^{-1}(K) \cap C$ and therefore $y' \in f(f^{-1}(K) \cap C)$.
Since X is non-empty, so is $f(X)$. Since Y is connected and $f(X)$ is a non-empty open and closed subset, $f(X)$ must equal Y.

(b) Since f is a local homeomorphism, it is in particular locally injective, so every $x \in f^{-1}(y)$ has a neighborhood U such that $f^{-1}(y) \cap U = \{x\}$. It follows that $f^{-1}(y)$ is discrete, but it is also compact since f is proper and $\{y\}$ is compact. But a compact discrete space must be finite, by considering the open cover by singletons.

(c) Let $f^{-1}(y) = \{x_1, \dots, x_n\}$, as per part (a). Since f is a local homeomorphism, there are neighborhoods U_i' of x_i such that $f|_{U_i'} : U_i' \to f(U_i')$ are homeomorphisms and $f(U_i')$ are open in Y. Since X is Hausdorff, we can also find disjoint neighborhoods U_{ij} and U_{ji} of x_i and x_j respectively, for each $j \neq i$. Let $U_{x_i} = U_i := U_i' \cap \bigcap_{j \neq i} U_{ij}$, which is a finite intersection of neighborhoods of x_i and therefore is itself a neighborhood of x_i. Since $U_i \subset U_i'$, the restriction $f|_{U_i}$ is also a homeomorphism onto its image. Further, since $U_i \subset U_{ij}$ and $U_j \subset U_{ji}$, it follows that $U_i \cap U_j = \varnothing$ for $i \neq j$.

(d) By the above construction, $V_i = f(U_i)$ are open neighborhoods of y and $f|_{U_i} : U_i \to V_i$ are homeomorphisms. Let $V = (\bigcap V_i) \setminus f(X \setminus \bigcup U_i)$. From part (a), f is a closed map, so $f(X \setminus \bigcup U_i)$ is closed. Since f is also an open map (also from part (a)), $\bigcap V_i$ is open and therefore so is V. Since $x_i \in U_i$, we get $y \in \bigcap V_i$ and since $f^{-1}(y) \subset \bigcup U_i$ we get $y \in V$. In general, if $x \notin \bigcup U_i$ then $f(x) \notin V$ so, by taking the contrapositive, $f^{-1}(V) \subset \bigcup U_i$ and therefore $f^{-1}(V) = \bigcup (f^{-1}(V) \cap U_i)$. Since we've constructed U_i to be pairwise disjoint, so are $f^{-1}(V) \cap U_i$. Finally, since $V \subset V_i$ and $f|_{U_i} : U_i \to V_i$ is a homeomorphism, so is its restriction $f|_{f^{-1}(V) \cap U_i} : f^{-1}(V) \cap U_i \to V$. This is precisely what it means for V to be evenly covered.

(5) Prove the theorem of *Existence of the Universal Covering Space*. More precisely, fix a point x_0 in a locally simply connected topological space X, let $\tilde{X}$ be the set of path classes starting at x_0 and define a map $q : \tilde{X} \to X$ by $q([f]) = f(1)$. Then:

  (a) [2 pts] Define a topology on $\tilde{X}$.
  Assume it now to be known that $\tilde{X}$ is path connected.

  (b) [2 pts] Show that q is a covering map.

  (c) [2 pts] Show that $\tilde{X}$ is simply connected.

Solution: This is part of Theorem 11.43 from the book, and its proof can be found there.