MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonas Bergström

Tentamensskrivning i
Topologi MM7052
7,5 hp
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(1) Let $\mathbb{Z}^+ = \{k \in \mathbb{Z} : k > 0\}$. An arithmetic progression is a set of integers of the form
$$a + b\mathbb{Z} = \{a + bk : k \in \mathbb{Z}\},$$
where $a, b \in \mathbb{Z}^+$. Put $V_{a,b} = (a + b\mathbb{Z}) \cap \mathbb{Z}^+$.
- (a) [1 pt] Show that $V_{a,b} \cap V_{c,d} \neq \emptyset$ if and only if $a \equiv_{\gcd(b,d)} c$.
- (b) [2 pts] Show that the sets $V_{a,b}$ for all $a, b \in \mathbb{Z}^+$ with $\gcd(a, b) = 1$ form a basis for a topology T on $\mathbb{Z}^+$.
- (c) [1 pt] Show that kb is contained in the closure of $V_{a,b}$ for any $k \in \mathbb{Z}^+$.
- (d) [1 pt] Show that $(\mathbb{Z}^+, T)$ is connected.
- (e) [1 pt] Show that $(\mathbb{Z}^+, T)$ is not compact.
  Hint: Consider sets $V_{p-1,p}$ for p prime.

(2) Let $X = \mathbb{R}^{\mathbb{N}}$ be the set of infinite sequences of real numbers. Consider the box topology T on X generated by $U_1 \times U_2 \times \ldots$ such that each U_i is open in $\mathbb{R}$. Recall that the product topology T' on X is generated by $U_1 \times U_2 \times \ldots$ such that each U_i is open and all but finitely many U_i are equal to $\mathbb{R}$.
- (a) [1 pt] Show that $T' \subset T$ but $T \not\subset T'$.
- (b) [1 pt] Show that the map $f : \mathbb{R} \to X$ defined by $x \mapsto (x_n)$, with $x_n = x$ for all n, is not continuous in the topology T.
- (c) [2 pts] Show that (X, T') is path connected.
- (d) [2 pts] Show that the set of bounded sequences is both open and closed and hence that (X, T) is not connected.
  Hint: For any $(a_n) \in X$, $(a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \ldots$ consists of either only bounded sequences or only unbounded sequences.

(3) Consider $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. For $m \geq 1$, let $f_m : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $z \mapsto z^m$. Let D be a closed 2-cell and consider f_m as a map from the boundary of D to $\mathbb{S}^1$. Let X be the wedge sum $(\mathbb{S}^1 \cup_{f_2} D) \vee (\mathbb{S}^1 \cup_{f_2} D)$. (You can assume all base points appearing in this exercise to be nondegenerate.)
- (a) [2 pts] Compute the fundamental group $G = \pi_1(X)$.
- (b) [1 pt] Prove or disprove that G contains elements of infinite order.
- (c) [2 pts] Compute the abelianization of the group G.
- (d) [1 pt] Compute the fundamental group of $(\mathbb{S}^1 \cup_{f_2} D) \cup_{\tilde{f}_3} D$, where $\tilde{f}_3$ denotes the composition of $f_3 : \partial D \to \mathbb{S}^1$ with the natural map $\mathbb{S}^1 \to \mathbb{S}^1 \cup_{f_2} D$.

(4) Consider $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$.
- (a) [2 pts] For a positive integer $n \in \mathbb{Z}$, verify that the action of $\mathbb{Z}/n$ on $\mathbb{S}^3$ by $[k] \cdot (z, w) = (e^{2\pi ik/n}z, e^{2\pi ik/n}w)$ is a covering space action.
- (b) [1 pt] Let L_n be the quotient $\mathbb{S}^3/(\mathbb{Z}/n)$. Compute $\pi_1(L_n)$.
- (c) [1 pt] If m divides n, find a covering map $\psi_{m,n} : L_m \to L_n$.
- (d) [2 pts] Show that every continuous map $f : L_n \to \mathbb{T}^2$ (where $\mathbb{T}^2$ is the torus), is homotopic to a constant map.
  Hint: Use a lifting to the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$.

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(5) [6 pts] Prove the following theorems:

Theorem 1. *Every closed subset of a compact space is compact.*

Theorem 2. *Every compact subset of a Hausdorff space is closed.*

Theorem 3. *If F is a continuous map from a compact space to a Hausdorff space then F is a closed map.*