MATEMATISKA INSTITUTIONEN
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Avd. Matematik
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Tentamensskrivning i
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(1) Let $\mathbb{Z}^+ = \{k \in \mathbb{Z} : k > 0\}$. An arithmetic progression is a set of integers of the form

$$a + b\mathbb{Z} = \{a + bk : k \in \mathbb{Z}\},$$

where $a, b \in \mathbb{Z}^+$. Put $V_{a,b} = (a + b\mathbb{Z}) \cap \mathbb{Z}^+$.

(a) [1 pt] Show that $V_{a,b} \cap V_{c,d} \neq \emptyset$ if and only if $a \equiv_{\gcd(b,d)} c$.

(b) [2 pts] Show that the sets $V_{a,b}$ for all $a, b \in \mathbb{Z}^+$ with $\gcd(a, b) = 1$ form a basis for a topology T on $\mathbb{Z}^+$.

(c) [1 pt] Show that kb is contained in the closure of $V_{a,b}$ for any $k \in \mathbb{Z}^+$.

(d) [1 pt] Show that $(\mathbb{Z}^+, T)$ is connected.

(e) [1 pt] Show that $(\mathbb{Z}^+, T)$ is not compact.
Hint: Consider sets $V_{p-1,p}$ for p prime.

Solution:

(a) For brevity of notation, let $g = \gcd(b, d)$.
Suppose $n \in V_{a,b} \cap V_{c,d}$. Then $g \mid b \mid n - a$, so $n \equiv_g a$. Similarly $n \equiv_g c$. Combining, we get $a \equiv_g c$.
Conversely, let $a \equiv_g c$. Since g is the gcd of b and d, there exist $x, y \in \mathbb{Z}$ such that $bx+dy = g$. Multiplying by the integer $\frac{a-c}{g}$, we get $b\frac{a-c}{g}x+d\frac{a-c}{g}y = a-c$. Rearranging, $a - b\frac{a-c}{g}x = c + d\frac{a-c}{g}y$. Call this integer $n' \in (a + b\mathbb{Z}) \cap (c + d\mathbb{Z})$. A priori n' may not be > 0, but for any $k \in \mathbb{Z}$, the translate $n = n' + kbd$ is in the same intersection, and for k large enough $n > 0$, so $n \in V_{a,b} \cap V_{c,d}$.

(b) First, for any $n \in \mathbb{Z}^+$, we have $n + 1 > 1$, so there is some prime $p|n + 1$, and therefore $p|n - (p - 1)$. In other words, every $n \in V_{p-1,p}$ for some prime p (note that $\gcd(p - 1, p) = 1$). Therefore the collection of all $V_{a,b}$ (with $\gcd(a, b) = 1$) all covers $\mathbb{Z}^+$.
Now, by part (a), $V_{a,b} \cap V_{c,d}$ is either empty or $a \equiv_{\gcd(b,d)} c$. In the latter case, let $n \in V_{a,b} \cap V_{c,d}$. Note that $V_{a,b} = V_{n,b}$ and $V_{c,d} = V_{n,d}$, so assume $a = c = n$. We claim that $V_{n,b} \cap V_{n,d} = V_{n,\mathrm{lcm}(b,d)}$, which finishes the check that the collection of $V_{a,b}$ form a basis for a topology. It suffices to verify that $(n + b\mathbb{Z}) \cap (n + d\mathbb{Z}) = n + \mathrm{lcm}(b, d)\mathbb{Z}$. But this is equivalent to $b, d \mid m - n$ if and only if $\mathrm{lcm}(b, d) \mid m - n$, which is a basic property (or perhaps the definition) of the lcm.

(c) We want to show that any neighborhood of kb intersects $V_{a,b}$. It suffices to check this for basic neighborhoods, so consider c, d with $\gcd(c, d) = 1$ such that $kb \in V_{c,d}$. Since $kb \equiv_d c$, it follows that $\gcd(kb, d) = \gcd(c, d) = 1$ and therefore $\gcd(b, d) = 1$. Now $a \equiv_1 c$ is automatic so, by part (a), $V_{c,d}$ indeed intersects $V_{a,b}$.

(d) Consider any two non-empty sets $A, B \subset \mathbb{Z}^+$ which are both open and closed. Therefore there is some $V_{m,a} \subset A$ and some $V_{n,b} \subset B$ with $\gcd(m, a) = \gcd(n, b) = 1$. By part (c), ab is in the closure of both of these and therefore in $A \cap B$. In particular $A \cap B \neq \varnothing$, so they cannot form a disconnection.

(e) As in the hint, consider the collection of basic open sets $V_{p-1,p}$ for p prime. We already showed in part (b) that these cover $\mathbb{Z}^+$ so it suffices to show that no finite subcollection covers. Note that a positive integer n belongs to $V_{p-1,p}$ iff $p \mid n + 1$. So for any finite set of primes S, if we take $n = q - 1$ for some prime $q \notin S$, then n is not in any $V_{p-1,p}$ for $p \in S$.

(2) Let $X = \mathbb{R}^{\mathbb{N}}$ be the set of infinite sequences of real numbers. Consider the box topology T on X generated by $U_1 \times U_2 \times \ldots$ such that each U_i is open in $\mathbb{R}$.

Recall that the product topology T' on X is generated by $U_1 \times U_2 \times \dots$ such that each U_i is open and all but finitely many U_i are equal to $\mathbb{R}$.

(a) [1 pt] Show that $T' \subset T$ but $T \not\subset T'$.

(b) [1 pt] Show that the map $f : \mathbb{R} \to X$ defined by $x \mapsto (x_n)$, with $x_n = x$ for all n, is not continuous in the topology T.

(c) [2 pts] Show that (X, T') is path connected.

(d) [2 pts] Show that the set of bounded sequences is both open and closed and hence that (X, T) is not connected.
Hint: For any $(a_n) \in X$, $(a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \dots$ consists of either only bounded sequences or only unbounded sequences.

Solution:

(a) Since the given basis of T' is contained in that of T, it follows that $T' \subset T$.
Any $\varnothing \neq V \in T'$ contains some non-empty basic open set $U = U_1 \times U_2 \times \cdots$ with $U_i = \mathbb{R}$ for all but finitely many i. Denoting by π_i the ith projection $X \to \mathbb{R}$, we get $\pi_i(V) \supset \pi_i(U) = U_i = \mathbb{R}$ for all but finitely many i. Therefore $W = (0,1) \times (0,1) \times \cdots$ cannot be in T' since $\pi_i(W) = (0,1)$ for each i. This is a basic open set of T, so $W \in T \setminus T'$.

(b) Let $U_n = (-1/n, 1/n)$ and consider the open neighborhood $U = U_1 \times U_2 \times \cdots$ of $0 \in X$. Then $f^{-1}(U) = \bigcap_{n \geq 1} U_n = \{0\}$ is not open in $\mathbb{R}$, so f is not continuous.

(c) Let $(a_n), (b_n)$ be any two points of X. Let $f(t) = (ta_n + (1-t)b_n)_{n \geq 1}$ define a function $f : [0,1] \to X$. Evidently $f(0) = b$, $f(1) = a$ and each $f_i := \pi_i \circ f : [0,1] \to \mathbb{R}$ is continuous. Therefore by the characteristic property of the product topology, f is a path from (b_n) to (a_n) in the topology T'. Since these points were arbitrary, (X, T') is path connected.

(d) Following the hint, note that $U_a = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots$ is a neighborhood of the sequence $a = (a_n)_{n \geq 1} \in X$. Further, if a is bounded, ie there is some $C > 0$ such that each $|a_n| < C$, then for any $b = (b_n) \in U_a$, each $|b_n| < C + 1$, so b is bounded as well. Conversely if a is unbounded, then for any $b \in U_a$ and $C > 0$ there is some n such that $|a_n| > C + 1$ and hence $|b_n| > C$, so b is unbounded as well. In other words, writing $B \subset X$ for the set of bounded sequences, if $a \in B$ then $U_a \subset B$ and if $a \notin B$ then $U_a \subset X \setminus B$. It follows that both B and $X \setminus B$ are open, thereby providing a disconnection.

(3) Consider $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. For $m \geq 1$, let $f_m : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $z \mapsto z^m$. Let D be a closed 2-cell and consider f_m as a map from the boundary of D to $\mathbb{S}^1$. Let X be the wedge sum $(\mathbb{S}^1 \cup_{f_2} D) \vee (\mathbb{S}^1 \cup_{f_2} D)$. (You can assume all base points appearing in this exercise to be nondegenerate.)

(a) [2 pts] Compute the fundamental group $G = \pi_1(X)$.

(b) [1 pt] Prove or disprove that G contains elements of infinite order.

(c) [2 pts] Compute the abelianization of the group G.

(d) [1 pt] Compute the fundamental group of $(\mathbb{S}^1 \cup_{f_2} D) \cup_{\tilde{f}_3} D$, where $\tilde{f}_3$ denotes the composition of $f_3 : \partial D \to \mathbb{S}^1$ with the natural map $\mathbb{S}^1 \to \mathbb{S}^1 \cup_{f_2} D$.

Solution:

(a) Since $\mathbb{S}^1$ is path connected, so is $\mathbb{S}^1 \cup_{f_2} D$ and hence X. Now, if a is the the generator of $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ then $\pi_1(\mathbb{S}^1 \cup_{f_2} D) \cong \langle a \mid a^2 \rangle \cong \mathbb{Z}/2$. In anticipation of part (d), let us also note that the map $\pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{S}^1 \cup_{f_2} D)$ induced by the natural map $\mathbb{S}^1 \to \mathbb{S}^1 \cup_{f_2} D$ can be taken to be the quotient $\langle a \rangle \to \langle a \mid a^2 \rangle = \langle a \rangle / \langle a^2 \rangle$.
Now since any basepoint in $(\mathbb{S}^1 \cup_{f_2} D)$ is nondegenerate, it follows that $\pi_1(X) \cong \mathbb{Z}/2 * \mathbb{Z}/2$.

(b) By above, write $\pi_1(X) = \langle a \rangle / \langle a^2 \rangle * \langle b \rangle / \langle b^2 \rangle$. Then for each $n \geq 1$, the word $(ab)^n = abab \cdots ab$ is *reduced* in the free product, and therefore non-trivial. In other words, the element ab has infinite order.

(c) By part (a), we have $G = \pi_1(X) = \langle a, b \mid a^2, b^2 \rangle$. Consider the homomorphism $\tilde{\phi} : G \to \mathbb{Z}/2 \times \mathbb{Z}/2 =: H$ defined by $\tilde{\phi}(a) = (1,0)$ and $\tilde{\phi}(b) = (0,1)$. Then $\tilde{\phi}$ is well-defined since $2 \cdot (1,0) = (0,0)$ and $2 \cdot (0,1) = (0,0)$ in H and also surjective since $\{\phi(a), \phi(b)\}$ generates H. Since H is abelian, $\tilde{\phi}$ descends to a surjective homomorphism $\phi : G^{\text{ab}} \to H$.
By slight abuse of notation use a and b to also denote their respective images in G^{ab}. It suffices to show that there is a homomorphism $\psi : H \to G^{\text{ab}}$ such that $\psi(1,0) = a$ and $\psi(0,1) = b$, which will then be the inverse of ϕ. In G^{ab}, we have $a^2 = b^2 = (ab)^2 = 1$, $a(ab) = a^2 b = b$ and $b(ab) = ab^2 = a$. Then defining $\psi(0,0) = 1$ and $\psi(1,1) = ab$, and using the commutativity in G^{ab}, this verifies directly from the definition that ψ is a homomorphism (note that there are only six products of non-trivial elements of H up to commutativity).

(d) If ω is the generator of $\pi_1(\partial D) \cong \mathbb{Z}$ and a is the generator of $\pi_1(\mathbb{S}^1)$ then (up to a choice of signs) we have $(f_3)_*(\omega) = a^3$. Therefore, by what we explained in part (a), we have that $(\tilde{f_3})_*(\omega) = a^3 = a \in \langle a \rangle / \langle a^2 \rangle$. Therefore

$$\pi_1((\mathbb{S}^1 \cup_{f_2} D) \cup_{\tilde{f}_3} D) \cong \langle a \mid a^2, a \rangle = \langle a \mid a \rangle \cong \{1\},$$

the trivial group.

(4) Consider $\mathbb{S}^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$.
   (a) [2 pts] For a positive integer $n \in \mathbb{Z}$, verify that the action of $\mathbb{Z}/n$ on $\mathbb{S}^3$ by $[k] \cdot (z,w) = (e^{2\pi i k/n} z, e^{2\pi i k/n} w)$ is a covering space action.
   (b) [1 pt] Let L_n be the quotient $\mathbb{S}^3/(\mathbb{Z}/n)$. Compute $\pi_1(L_n)$.
   (c) [1 pt] If m divides n, find a covering map $\psi_{m,n} : L_m \to L_n$.
   (d) [2 pts] Show that every continuous map $f : L_n \to \mathbb{T}^2$ (where $\mathbb{T}^2$ is the torus), is homotopic to a constant map.
   Hint: Use a lifting to the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$.

Solution: We use the notation $\zeta_n = e^{2\pi/n}$ for positive integers n. Note that ζ_n has (multiplicative) order n.

(a) For each $(z,w) \in \mathbb{S}^3$, at least one of z and w is non-zero, so $[k] \cdot (z,w)$ determines $\zeta_n^k = e^{2\pi k/n}$ and therefore $[k] \in \mathbb{Z}/n$. In other words the action is free.
For each $[k] \in \mathbb{Z}/n$, the map $(z,w) \mapsto \zeta_n^k \cdot (z,w)$ is a linear map $\mathbb{C}^2 \to \mathbb{C}^2$, in particular continuous. Therefore its restriction to S^3 is continuous as well. Note that $|\zeta_n^k z|^2 + |\zeta_n^k w|^2 = |z|^2 + |w|^2$, so this map indeed takes S^3 to itself.
It remains to note that any continuous free action of a finite group G on a Hausdorff space X is a covering space action: for any $x \in X$, and $g \neq 1$, if $U_g \ni x$ and $V_g \ni g \cdot x$ are disjoint open sets then $U = \bigcap_{g \neq 1} U_g \cap g^{-1} V_g$ has the property that $gU \cap U = \varnothing$ for $g \neq 1$. Explicitly, f $y, z \in U$ and $g \neq 1$ then $y \in g^{-1} V_g$ so $g \cdot y \in V_g$ cannot equal $z \in U_g$.

(b) Since $\mathbb{S}^3$ is simply-connected and the action of $\mathbb{Z}/n$ is a covering space action, $\pi_1(L_n) = \pi_1(\mathbb{S}^3/\mathbb{Z}/n) \cong \mathbb{Z}/n$.

(c) Since $\mathbb{S}^3 \to L_n$ is a universal covering map of the connected space L_n, other connected covers of L_n correspond (up to isomorphism) to normal subgroups of $\pi_1(L_n) = \mathbb{Z}/n$ (ie to subgroups, since this group is abelian). Now, under the assumption that m divides n, the subgroup $(n/m)(\mathbb{Z}/n)$ is isomorphic to $\mathbb{Z}/m$ and the action of $[k] \in \mathbb{Z}/m$ on $\mathbb{S}^3$ is by that of $[kn/m] \in \mathbb{Z}/n$, ie $[k] \cdot (z,w) = \zeta_n^{kn/m} \cdot (z,w) = \zeta_m^k (z,w)$. Therefore by the classification of covering spaces, we get a covering map $L_m := \mathbb{S}^3/(\mathbb{Z}/m) \to L_n$.

(d) Let $f : L_n \to \mathbb{T}^2$ be continuous and choose $p \in L_n$. Then the induced map $f_* : \pi_1(L_n, p) \to \pi_1(\mathbb{T}^2, f(p))$ is a group homomorphism $\mathbb{Z}/n \to \mathbb{Z}$ and therefore must be trivial (the only element of finite order in $\mathbb{Z}$ is the identity). It follows that f has a lift $\tilde{f} : L_n \to \mathbb{R}^2$ along the universal cover $\pi : \mathbb{R}^2 \to \mathbb{T}^2$.
But $\mathbb{R}^2$ is contractible, so $\tilde{f}$ must be nullhomotopic, ie $\tilde{f} \simeq c$ for some constant map c. Therefore $f = \pi \circ \tilde{f} \simeq \pi \circ c$, which is also a constant map.

(5) [6 pts] Prove the following theorems:

Theorem 1. *Every closed subset of a compact space is compact.*

Theorem 2. *Every compact subset of a Hausdorff space is closed.*

Theorem 3. *If F is a continuous map from a compact space to a Hausdorff space then F is a closed map.*

Solution: These theorems are Propositions 4.36(a), 4.36(b) and Lemma 4.50, respectively, from the book, and their proofs can be found there.