Department of Mathematics
Stockholm University

MM 7052 - Topology

Fall 2025 - final exam

Examiner: Gregory Arone
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- **No** use of textbook, notes, or calculators is allowed.
- Unless told otherwise, you may quote results that are proved in the textbook or were covered in class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.

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1. Let (X, d) be a metric space. Recall that a subset $A \subset X$ is called bounded if it is contained in some ball of radius $R < \infty$. We define a new topology $\mathcal{T}$ on X as follows: a subset $A \subset X$ belongs to $\mathcal{T}$ if either A is empty or $X \setminus A$ is bounded.
   (a) (2 points) Prove that $\mathcal{T}$ really is a topology.
   (b) (1 point) Suppose X is itself an unbounded metric space. Is $(X, \mathcal{T})$ connected?
   (c) (2 points) Let $\mathbb{R}$ be the real line, with the usual metric. Is $(\mathbb{R}, \mathcal{T})$ compact?

2. (5 points) A topological space X is called *regular* if for every closed subset $A \subset X$, and every point $x \in X \setminus A$, there exists disjoint open subsets $U, V \subset X$ such that $A \subset U$ and $x \in V$.

   Prove that a locally compact Hausdorff space is regular. For partial credit, you may prove that a compact Hausdorff space is regular. A correct proof of this will be worth 3.5 points out of 5.

3. In this question, let us say that a function $f\colon X \to Y$ is *good* if it is a surjective continuous map between Hausdorff spaces.
   (a) (2 points) Show an example of a good function f that is open, but not closed, or show that such an f does not exist.
   (b) (3 points) Show an example of a good function f that is closed, but not open, or show that such an f does not exist.

4. (5 points) Suppose $f\colon X \to Y$ is a homotopy equivalence. Prove that for any choice of point $x_0 \in X$, f induces an isomorphism $f_*\colon \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, f(x_0))$. Suggestion: first prove that if $f, g\colon X \to Y$ are homotopic maps, then f_* and g_* are related by a certain isomorphism.

5. In this problem, let $B = \mathbb{R}P^2 \times \mathbb{R}P^2$
   (a) (2 points) Describe the universal cover of B.
   (b) (3 points) Determine the number of connected covering spaces of B. More explicitly, the question is the following: how many different covering maps $p\colon E \to B$ are there, up to isomorphism, where E is a path-connected space? Describe explicitly (or as explicitly as you can) the spaces E that occur as connected covers of B.

6. Let $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the closed unit disk in the complex plane, and let $S^1 \subset D^2$ be the unit circle, identified with the boundary of D^2.
   (a) (1 point) Describe the fundamental group of $D^2 \times S^1$. (This space is called the solid torus).
   (b) (1 point) Prove that the inclusion $S^1 \times S^1 \hookrightarrow D^2 \times S^1$ does not have a retraction.

(c) (3 points) Let p be a positive integer. Let X be the quotient of disjoint union of two copies of $D^2 \times S^1$ by the relation that identifies every point $(u, v) \in S^1 \times S^1$ on the boundary of the first solid torus with the point $(u, u^p v)$ on the boundary of the second solid torus. Give an explicit presentation of $\pi_1(X)$ and show that it is isomorphic to a familiar group.

Note: For this exercise you are allowed to argue a bit more intuitively and omit some technical details. But do try to make your reasoning as clear as possible.