MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Gregory Arone

Tentamensskrivning i
Topologi, MM8002/SF2721
7.5 hp
2019-12-18

(1) Consider the following family of subsets of $\mathbb{R}$:

$$\mathcal{T} := \{U \subseteq \mathbb{Z}\} \cup \{U \mid \mathbb{Z}\backslash(U \cap \mathbb{Z}) \text{ is finite}\}.$$

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$ which is neither coarser nor finer than the Euclidean topology.
(b) Determine interior, closure, exterior, and boundary of $\mathbb{Z}$ in the topology $\mathcal{T}$.
(c) Say for which real number a the singlet $\{a\}$ is open in the topology $\mathcal{T}$.
(d) Say for which real number a the singlet $\{a\}$ is closed in the topology $\mathcal{T}$. Determine if the topological space $(\mathbb{R}, \mathcal{T})$ is connected.
(e) Determine whether the topological space $(\mathbb{R}, \mathcal{T})$ is compact.
(f) Determine whether the topological space $(\mathbb{R}, \mathcal{T})$ is Hausdorff (T_2). [30 points]

Solution:
(a) We first show that $\mathcal{T}$ is a topology. As $\mathbb{Z}\backslash(\mathbb{R} \cap \mathbb{Z}) = \emptyset$ we have that clearly $\mathbb{R} \in \mathcal{T}$. At the same way $\emptyset \subseteq \mathbb{Z}$, so also the empty set is open. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ a family of sets in $\mathcal{T}$, then we have

$$\bigcup U_\alpha = \left(\bigcup_{U_\alpha \subseteq \mathbb{Z}} U_\alpha \right) \cup \left(\bigcup_{U_\alpha \not\subseteq \mathbb{Z}} U_\alpha \right) = U \cup V.$$

If V is empty we have that $\bigcup U_\alpha \subseteq \mathbb{Z}$, and therefore it is open. Otherwise there is α_0 such that $U_{\alpha_0} \not\subseteq \mathbb{Z}$) and we have that

$$\mathbb{Z}\backslash(\bigcup U_\alpha) = \mathbb{Z}\backslash((U \cup V) \cap \mathbb{Z} \cap \mathbb{Z}) \subseteq \mathbb{Z}\backslash(U_{\alpha_0} \cap \mathbb{Z}).$$

The latter set is finite by the definition of $\mathcal{T}$, so we have that this $\bigcup U_\alpha$ is open. Thus we need just to show that $\mathcal{T}$ is closed with respect of taking finite intersection. It is enough to show that the intersection of two elements of $\mathcal{T}$ is open, the more general case will easily follow by induction. So let U and V in $\mathcal{T}$. If one between U and V is a subset of $\mathbb{Z}$, we have that $U \cap V$ is a subset of $\mathbb{Z}$ and hence open. Thus we can suppose that neither U nor V are subsets of $\mathcal{T}$. But in this case, by De Morgan Law we have

$$\mathbb{Z}\backslash(U \cap V \cap \mathbb{Z}) = (\mathbb{Z}\backslash(U \cap \mathbb{Z})) \cup (\mathbb{Z}\backslash(V \cap \mathbb{Z})).$$

By the definiton of $\mathcal{T}$ the latter set is a union of finite set and so we get that $\mathbb{Z}\backslash(U \cap V \cap \mathbb{Z})$ is finite and therefore $U \cap V$ is open.

Now we compare $\mathcal{T}$ with the Euclidean topology. Let n be any integer, then $\{n\} \subseteq \mathbb{R}$ is open with the topology $\mathcal{T}$ but not with the Euclidean topology, so the Euclidean topology is not finer than $\mathcal{T}$. Conversely, consider the interval $I = (\frac{1}{2}, \frac{3}{2})$. This is clearly open with the Euclidean topology but $I \not\subseteq \mathbb{Z}$ and $\mathbb{Z}\backslash(I \cap \mathbb{Z}) = \mathbb{Z}\backslash\{1\}$ is clearly not finite. So I is not in $\mathcal{T}$ and we conclude that the Euclidean topology is not finer than $\mathcal{T}$.
(b) As $\mathbb{Z}$ is a subset of itself we have that clearly $\mathbb{Z}$ is open in $\mathcal{T}$. In particular we get that $\operatorname{Int} \mathbb{Z} = \mathbb{Z}$. Note that from the definition of $\mathcal{T}$ every non empty open set intersect $\mathbb{Z}$ because they are either non empty subsets of $\mathbb{Z}$ or $U \cap \mathbb{Z}$ must be infinitely countable. Let C be a closed set containing $\mathbb{Z}$, its complementary $U := \mathbb{R}\backslash\mathbb{Z}$ is an open set that does not intersect $\mathbb{Z}$.

We deduce that $U = \emptyset$ and $C = \mathbb{R}$. So the closure of $\mathbb{Z}$ the whole $\mathbb{R}$. As $\operatorname{Ext}\mathbb{Z} = \mathbb{R}\backslash\overline{\mathbb{Z}}$, we deduce that $\operatorname{Ext}\mathbb{Z} = \emptyset$. In addition we have that $\partial\mathbb{Z} = \mathbb{R}\backslash(\operatorname{Int}(\mathbb{Z}) \cup \operatorname{Ext}(\mathbb{Z}) = \mathbb{R}\backslash\mathbb{Z}$.
(c) If a is an integer we have that $\{a\} \subseteq \mathbb{Z}$ and so it is open. Conversely if a is not an integer then $\{a\}$ is clearly not a subset of $\mathbb{Z}$. In addition we have that $\mathbb{Z}\backslash(\{a\} \cap \mathbb{Z}) = \mathbb{Z}\backslash\emptyset = \mathbb{Z}$, so $\{a\}$ is not open. We conclude that $\{a\}$ is open if, and only if, $a \in \mathbb{Z}$. (d) All singlets are closed: given $a \in \mathbb{R}$, we have that $(\mathbb{R}\backslash\{a\}) \cap \mathbb{Z}$ is either the whole $\mathbb{Z}$ - if a is not an integer - or $\mathbb{Z}\backslash\{a\}$. In any case we have that $\mathbb{Z}\backslash((\mathbb{R}\backslash\{a\}) \cap \mathbb{Z})$ is finite. We deduce that $\mathbb{R}$ with this topology cannot be connected since sets of the form $\{a\}$ with a an integer are non empty, and clopen. (e) Let $a \notin \mathbb{Z}$, then we have that $U_x := \mathbb{Z} \cap \{x\}$ is open as $\mathbb{Z}\backslash(U_x \cap \mathbb{Z})$ is empty. If one consider the open cover $\{U_x\}_{x\in\mathbb{R}\backslash\mathbb{Z}}$ it is easy to see that no subcover of its can cover $\mathbb{R}$.
(f) This space is not Hausdorff. In fact let a and b in $\mathbb{R}\backslash\mathbb{Z}$ and suppose that there are two open sets U_a and U_b with $a \in U_a$ and $b \in U_b$ such that $U_a \cap U_b = \emptyset$. Let $V_a := U_a \cap \mathbb{Z}$, and similarly $V_b := U_b \cap \mathbb{Z}$. Then $V_a \cap V_b$ is empty and $V_a \subseteq \mathbb{Z}\backslash V_b$, which, by the definition of $\mathcal{T}$ is finite. But then we would have that $\mathbb{Z}\backslash V_a$ cannot be finite, contradicting that U_a is open in the given topology.

(2) Consider the real line $\mathbb{R}$ and let

$$\mathcal{B} := \{[a, b) \mid a, b \in \mathbb{R},\ a < b\}.$$

(a) Show that $\mathcal{B}$ is a basis for a topology on the real line.
(b) Determine if $(\mathbb{R}, +)$, where $\mathbb{R}$ is endowed with the topology generated by $\mathcal{B}$, is a topological group.

[20 points]

Solution:
(a) We want to show that $\mathcal{B}$ satisfies the basis criterion.

- The elements of $\mathcal{B}$ cover the real line: given a real number x we have that $[x, x+1)$ is in $\mathcal{B}$ and contains x. So the $\mathcal{B}$ satisfies the first requirement of the basis criterion.
- Let $[a_1, b_1)$ and $[a_2, b_2)$ two elements of $\mathcal{B}$. We want to show that if their intersection is non empty, then, for every $x \in [a_1, b_1) \cap [a_2, b_2)$ there is $B \in \mathcal{B}$ such that $x \in B \subseteq [a_1, b_1) \cap [a_2, b_2)$. If $[a_1, b_1) \cap [a_2, b_2) \neq \emptyset$ we have that $[a_1, b_1) \cap [a_2, b_2) = [\max\{a_1, a_2\}, \min\{b_1, b_2\})$ and clearly $\max\{a_1, a_2\} < \min\{b_1, b_2\})$ so we just take $B = [\max\{a_1, a_2\}, \min\{b_1, b_2\})$.

(b) The group $(\mathbb{R}, +)$ where $\mathbb{R}$ is endowed with the topology generated by $\mathcal{B}$ is not a topological group. If it were, we would have that the map $i : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto -x$ would need to be continuous. But $i^{-1}([a, b)) = (-a, -b]$ which is not open in the topology generated by $\mathcal{B}$. In fact, if a set of the form $(c, d]$ where to be open with respect to the topology in object, then we would also have that $\{d\} = (c, d] \cap [d, d+1)$ would also be open, and therefore could be written as union of elements of $\mathcal{B}$. As element of $\mathcal{B}$ have all the cardinality of continuous, it is not possible to write a non empty finite set as union of them.

(3) Let X_4 the four dimensional simplex, and consider Y its 2-skeleton.
(a) Determine the n-skeleton for Y for every n.
(b) Compute the Euler characteristic of Y.

[20 points]

Solution:
(a) We recall that $X_4 := \{\sum_{\{} i = 0\} t_i p_i\}$ where the p_i's are five affinely

independent points in some $\mathbb{R}^N$ and $t_i \in [0,1]$ such that $\sum t_i = 1$. It has 5 0-dimensional faces corresponding to the 5 points. It also has $10 = \binom{5}{2}$ 1-dimensional faces E_i with $i = 1, \dots, 10$, corresponding to simplices genereted by the subsets with two elements of $\{p_0, \dots, p_5\}$. Finally it also has $10 = \binom{5}{3}$ 2-dimensional faces S_j, corresponding to the simplices generated by the subset of cardinality 3 of $\{p_0, \dots, p_5\}$. Thus we have that

$$Y = \bigcup_{k=0}^{5} \{p_k\} \cup \bigcup_{i=0}^{10} E_i \cup \bigcup_{j=0}^{10} S_j.$$

For every $n \geq 2$ the n-skeleton of Y is just Y. The 0-skeleton of Y is

$$Y_0 = \bigcup_{k=0}^{5} \{p_k\}.$$

The 1-skeleton of Y is

$$Y_1 = \bigcup_{k=0}^{5} \{p_k\} \cup \bigcup_{i=0}^{10} E_i$$

(b) The Euler characteristic of Y is 5-10+10=5.

(4) Let $\mathbb{S}^1 \subseteq \mathbb{C}$ and consider the map $f : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $z \mapsto z^3$. Let D a closed 2-cell and consider f as a map from $\partial D \to \mathbb{S}^1$. Compute the fundamental group of $\mathbb{S}^1 \cup_f D$, the space obtained by attaching D to $\mathbb{S}^1$ along f. [10 points]

Solution:
Let $X := \mathbb{S}^1 \cap D$ let $p \in D \backslash \partial D$ viewed as a subspace of X (the equivalence relation used to construct X as a quotient is trivial restricted to $D \backslash \partial D$). Consider the following subsets of X: $U := D \backslash \partial D$ and $X \backslash \{p\}$. If $\pi : \mathbb{S}^1 \sqcup D \to X$ is the quotient map, we have that $\pi^{-1}(U) = i_D(D \backslash \partial D)$, where $i_D : D \hookrightarrow \mathbb{S}^1 \sqcup D$ is the natural inclusion in the disjoint union. This is an open set of the disjoint union and therefore, by the definition of quotient topology we have that this is open. At the same way we have that $\pi^{-1}(V) = \mathbb{S}^1 \sqcup D \backslash \{p\}$ which is also open in the disjoint union topology, and hence V is open too. Clearly $U \cap V \simeq \mathbb{B} \backslash \{0\}$ is path connected and we can write $X = U \cap V$. Therefore we can apply the SVK theorem in order to compute the fundamental group of X. We have that $\pi_1(U)$ is trivial has the interior of a two cell is contractible. We have that $X \backslash \{p\}$ is homotopically equivalent to $\mathbb{S}^1$ so $\pi_1(U) \simeq \mathbb{Z}$. By the SVK theorem we have that $\pi_1(X) \simeq \mathbb{Z}/N$, where N is the subgroup generated by $i_*(\gamma)$, where γ is the generator of $\pi_1(U \cap V) \simeq \mathbb{Z}$ and $i : U \cap V \hookrightarrow V$. Now we observe that due to the attaching map the γ loop revolving on single time clockwise around p is sent by i_* to a loop revolving 3 times around p. So we get that $\pi_1(X) \simeq \mathbb{Z}/3\mathbb{Z}$.
Another way to do this is to show that X is obtained from a triangle by cyclically identify the edges and use polygonal presentations.

(5) Consider the statement *The fundamental group is homotopy invariant.*
 (a) Spell out the precise mathematical meaning of the statement.
 (b) Prove the statement. [20 points]