# Solutions for the 2019-09-27 Topology exam.

## Exercise 1:

(a) We recall that a space has the discrete topology iff singlets $\{a\}$ are open

Let $X$ a space with the discrete topology and $S \subseteq X$ a subset endowed with the subspace topology then $s \in S$, $\{s\} = \{s\} \cap S$ as $\{s\}$ is open in $X$ $\{s\}$ is open in $S$ thus $S$ is discrete

(b) Let $X$ and $Y$ be discrete spaces. Then

$$B_X := \{\{x\} \mid x \in X\}$$

$$B_Y := \{\{y\} \mid y \in Y\}$$

are basis for the topology on $X$ an $Y$. Thus the product topology on $X \times Y$ is generate by

$$B_{X\times Y} := \{\{x\} \times \{y\} \mid x \in X,\ y \in Y\}$$

$$= \{\{(x,y)\} \mid x \in X,\ y \in Y\}$$

we deduce that $X \times Y$ is discrete

(c) Let $\pi: X \to Q$ be a quotient map and suppose that $X$ is discrete. We recall that by the definition of quotient topology

$V \subseteq Q$ is open iff $\pi^{-1}(V)$ is open in $X$

as any set is open in $X$ any set is open in $Q$ which therefore has the quotient topology

a1) Suppose now that $X$ is a trivial top space. Therefore $\mathcal{T}_X = \{\emptyset, X\}$
Let $S \subseteq X$, $\mathcal{T}_S = \{\emptyset \cap S, X \cap S\} = \{\emptyset, S\}$
thus $S$ is trivial.

b1) Let $X, Y$ trivial topological spaces, the product topology is generated by

$\{X \times Y\}$ and therefore is trivial

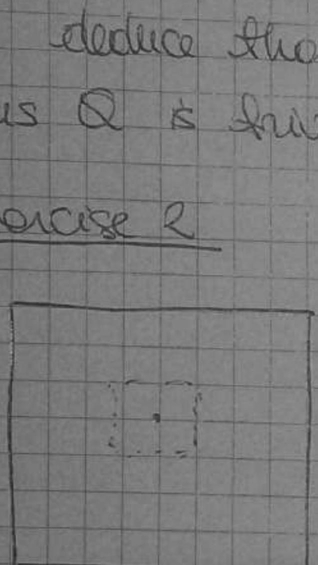
c1) Let $\pi : X \to Q$ a quotient map

- Let $V \subseteq Q$. $\pi^{-1}(V) = \emptyset$ iff $V = \emptyset$ as $\pi$ is surjective.

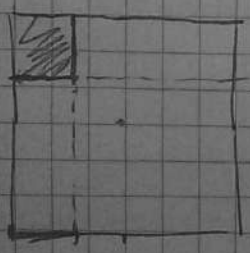
  $\pi^{-1}(V) = X \iff V = Q$

We deduce that $V$ is open in $Q$ iff $V = \emptyset$ or $V = Q$
thus $Q$ is trivial

Exercise 2



(a)



$X \setminus \left([-1, -\frac{1}{2}] \times [\frac{1}{2}, 1]\right) =$

$\underbrace{(-1, 1) \times (-1, \frac{1}{2})}_{\text{open}} \cup \underbrace{(-\frac{1}{2}, 1) \times (-1, 1)}_{\text{open}}$

As the union of opens is open we deduce that the complementary of $[-1, -\frac{1}{2}] \times [\frac{1}{2}, 1]$ is open. Therfore $[-1, -\frac{1}{2}] \times [\frac{1}{2}, 1]$ is closed

(b) as all the sets
$(a,b)\times(c,d)$ with $a,c<0$ and
$b,d>0$ contain $(0,0)$
we deduce immediately that $X$ is not $T_0$

(c) Again by construction all the open sets (not $\emptyset$) contain $(0,0)$ so there are not no $\emptyset$ open sets which are disjoint. We deduce that $X$ is connected

(d) as the sets $(a,b)\times(c,d)$ with $a,c<0$ and
$b,d>0$
are open in the Euclidean topology we deduce that this is finer than the given topology.
thus the identity map
$$id: I^2 \longrightarrow X \quad \text{is continuos}$$
On the other side $S=(-1,-\frac{1}{2})\times(-\frac{1}{2},-\frac{1}{2})$ is open in $I^2$ but not in $X$. If we consider
$$id: X \longrightarrow I^2 \quad \text{we have that}$$
$$id^{-1}(S) = S \quad \text{is not open hence } id \text{ is}$$
not continuous

(e) The property of being compact is stable when passing to a coarser topology. As the topology on $X$ is coarser than the Euclidean topology and $I$ with the Euclidean top. is compact (e.g. it is closed and Bounded, so we can use Heine Borel) we deduce that $X$ is compact

## ~~(E)~~xercise 3

(a) This is not a covering as it is not surjective

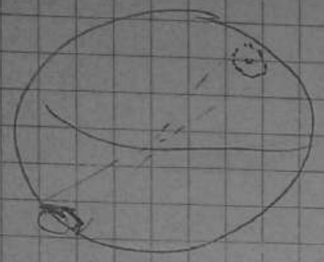
(b) This is not a covering as it is not surjective

(c) This is a covering. ~~the~~ As seen in class we can view this as a quotiant map

$$\mathbb{S}^2 / \mathbb{Z}/2\mathbb{Z}$$

Where $\mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{S}^2$ by $[1]x \longmapsto -x$

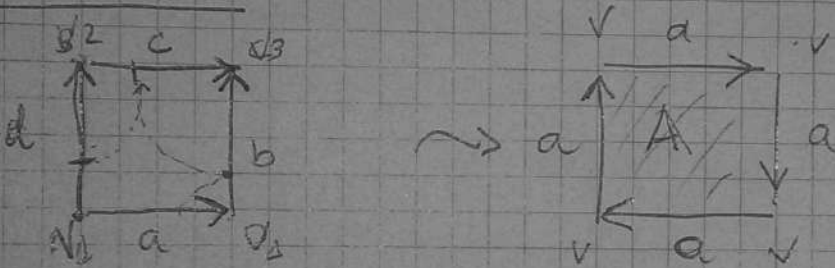
This action is totally discontinuos



as if we take a small enoug neighboun of $s \in \mathbb{S}^2$, $V$ we can have $V \cap -V = \emptyset$

Thus the map is a covering

## Exercise 4




0 Skeletan = $\{\{v\}\}$

1 Skeleton = $\{\{v\}\ \{a\}\}$

2 ———— = $\{\{v\}\ \{a\}, A\}$

As there is a unique vertex VK theorem state that

$$\pi_1(X) = \langle a \mid a^4 \rangle \cong \mathbb{Z}/4\mathbb{Z}$$

## Exercise 5

This is a theoretical question so we refer to the textbook