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MM7041–Topology
Final Exam

Feb 3, 2021
9:00-14:00
At home exam

Instructions: Work alone. You are allowed to use the textbook and the class notes. You can quote results from the textbook and from the class, but state clearly which result you are using. You are **not allowed** to search the internet for solutions or hints.

Justify all your answers with a proof or a counterexample. A simple Yes or No answer, even if correct, may get partial or no credit.

Problems have multiple parts. In some cases, later parts depend on earlier ones. Even if you could not do the earlier parts, you **are allowed** to use the results of the earlier parts in the later parts.

On the first page, please have the following:

- Name
- Social security number
- Write out and sign the following pledge: *On my honor as a student I have not received help or used inappropriate resources on this exam.*
- List the problems that you have attempted.

Start each problem on a new page (but it is not necessary to start each part of a problem on a new page). Write at the top of each page which problem it belongs to.

1. Let $X = [0, 2]$. Define a topology on X as follows: A set $U \subset X$ is open in X, if $U = \emptyset$ or if $[0, 1] \subset U$ (you do not need to prove that this really is a topology on X). Determine, with justification, whether X is
   (a) [2 pts] compact.
   (b) [1 pt] connected.
   (c) [1 pt] Hausdorff.
   (d) [1 pt] metrizable.
   (e) [2 pts] first countable.
   (f) [2 pts] second countable.

2. Let X be a topological space, and $A \subset X$ a subspace. Define the (not necessarily continuous) function $\chi_A \colon X \to \mathbb{R}$ as follows
$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$
   (a) [2 pts] Let $x \in X$. Prove that χ_A is continuous at x if and only if $x \notin \partial A$. (A function f is said to be continuous at a point x if x has an open neighborhood U such that the restriction of f to U is continuous).
   (b) [2 pts] Prove that χ_A is a continuous function if and only if A is both open and closed.

3. [4 pts] Let $\mathbb{R}_l$ be the real line, with the topology generated by the basis of sets of the form $[a, b)$. Let
$$X = [2,3] \cup \{-\frac{1}{n} \mid n = 1, 2, 3, \ldots\}.$$
Find the interior and the closure of X in $\mathbb{R}_l$.

4. [5 pts] Let X be a topological space satisfying the following property: For every two points $a, b \in X$, there exists a finite sequence of connected subspaces $C_1, \ldots, C_n$ of X such that $a \in C_1$, $b \in C_n$ and $C_i \cap C_{i+1} \neq \emptyset$ for all $i = 1, \ldots, n-1$. Prove that X is connected.

5. Let S^2 be a sphere, and let $a, b \in S^2$ be two distinct points. Let $S^2/\{a, b\}$ be the quotient space of S^2 by the relation that identifies a and b.
   (a) [3 pts] Describe a CW structure on $S^2/\{a, b\}$. Specify clearly how many cells of each dimension there are.
   (b) [1 pt] Compute the Euler characteristic of $S^2/\{a, b\}$.

6. Let S^2 be the two-dimensional sphere and let T be the torus. Let $x_1, x_2, x_3 \in S^2$ be three distinct points. Let $S^2 \setminus \{x_1, x_2, x_3\}$ denote the complement of three points in S^2 (do not confuse it with the quotient space of the previous question).
   (a) [2 pts] Use the van Kampen theorem to prove that $\pi_1(S^2 \setminus \{x_1, x_2, x_3\})$ is isomorphic to the free group on 2 generators.
   (b) [2 pts] Let $y \in T$ be any point. Prove that the spaces $T \setminus \{y\}$ and $S^2 \setminus \{x_1, x_2, x_3\}$ both contain the wedge sum (i.e., one-point union) $S^1 \vee S^1$ as a deformation retract. Conclude that they are homotopy equivalent.
   (c) [1 pt] The Jordan curve theorem asserts that the complement of any closed non-self-intersecting loop in $\mathbb{R}^2$ has two connected components. Assuming the Jordan curve theorem, prove that the spaces $T \setminus \{y\}$ and $S^2 \setminus \{x_1, x_2, x_3\}$ are not homeomorphic.