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MM7041–Topology
Final Exam

Jan 31, 2022
8:00-13:00

Instructions: Work alone. You are not allowed to use the textbook and the class notes. You can quote results that you learned in the class. Be sure to state clearly what results you are using.

Justify all your answers with a proof or a counterexample. A simple Yes or No answer, even if correct, may get partial or no credit.

Problems have multiple parts. In some cases, later parts depend on earlier ones. Even if you could not do the earlier parts, you **are allowed** to use the results of the earlier parts in the later parts.

1. Let τ be the following set of subsets of the real line $\mathbb{R}$: A set $U \subset \mathbb{R}$ is in τ if and only if one of the following conditions holds

   1. U is empty,
   2. $3 \in U$.

   You may take for granted that τ defines a topology on $\mathbb{R}$. Throughout this problem, let $\mathbb{R}$ denote the real line with the standard topology, and $\mathbb{R}_\tau$ denote the real line with the topology τ. Remember to justify your answers.

   (a) [1 pt] Is $\mathbb{R}_\tau$ compact?

   (b) [2 pts] Is $\mathbb{R}_\tau$ path-connected?

   (c) [1 pt] Is $\mathbb{R}_\tau$ first-countable?

   (d) [2 pts] Is $\mathbb{R}_\tau$ second-countable?

   (e) [2 pts] Describe all the continuous maps $\mathbb{R}_\tau \to \mathbb{R}$.

2. Let $f\colon X \to Y$ be a map (it goes without saying that a "map" is continuous). Recall that a retraction of f is a map $r\colon Y \to X$ such that $r \circ f = 1_X$.

   (a) [3 pts] Suppose f has a retraction, and Y is Hausdorff. Prove that that $f(X)$ is a closed subset of Y.

   (b) [2 pts] Under the same conditions as in part (a), prove that f is a closed map. Remember: you are allowed to use the conclusion of part (a) even if you did not do part (a).

   (c) [1 pt] Show that if Y is not Hausdorff, then the conclusion of part (a) is not necessarily true.

3. [4 pts] Let X be a space and $A \subset X$ a subspace. Suppose $C \subset X$ is a connected subspace, such that $C \cap A$ and $C \cap (X \setminus A)$ are both non-empty. Prove that $C \cap \partial A$ is non-empty. Here $\partial A = \bar{A} \cap \overline{X \setminus A}$ is the boundary of A.

4. Let X be a locally compact Hausdorff space.

   (a) [2 pts] Prove that a closed subset of X is locally compact.

   (b) [3 pts] Prove that an open subset of X is locally compact.

   (c) [1 pt] Show an example of a subspace of a locally compact Hausdorff space that is not locally compact.

5. (a) [2 pts] Let Prove that every map from $\mathbb{R}P^2$ to S^1 is homotopic to a constant map.
   (b) [2 pts] Give an example of two pointed spaces X and Y, and two pointed maps $f, g\colon X \to Y$, such that f and g are homotopic, but not pointed homotopic.

6. [3 pts] Let $X \subset \mathbb{R}^3$ be the space consisting of the four points $(0,0,0), (0,0,1), (0,1,0), (1,0,0)$, and the six line segments connecting them. In other words, X consists of the edges of a tetrahedron. What is the fundamental group of X?