Department of Mathematics
Stockholm University

MM7033 – Advanced Algebra

Fall 2023 – Final exam

Examiner: Gregory Arone
Date: December 14, 2023
Time: 14:00–19:00

- **No** use of textbook, notes, or calculators is allowed.
- Unless told otherwise, you may quote results that were proved in class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.
- In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts

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1. (a) (2 points) Determine the degree of the splitting field of $x^2 + x + 1$ over $\mathbb{F}_2$.
   (b) (3 points) Determine the degree of the splitting field of $x^4 - 2$ over $\mathbb{Q}$.
2. (5 points) Consider the following algebraic subsets over the complex numbers

$$X_1 = \mathcal{Z}(y - x^2) \subset \mathbb{A}^2, \qquad X_2 = \mathcal{Z}(xy - 1) \subset \mathbb{A}^2, \qquad X_3 = \mathcal{Z}(y^2 + x^2) \subset \mathbb{A}^2$$

   Determine whether or not some of the X_i's are isomorphic as algebraic subsets.
3. Let $R = \mathbb{Z}[x]/(x^2 - 1)$. As usual, given a polynomial $p(x)$ we denote by (p) or $(p(x))$ the ideal of R generated by the image of p in R. Note that there is an exact sequence of R-modules

$$0 \longrightarrow (x - 1) \longrightarrow R \longrightarrow R/(x - 1) \longrightarrow 0.$$

   (a) (1 point) Show that there is an isomorphism of R-modules: $(x + 1) \cong R/(x - 1)$.
   (b) (2 points) Is the sequence split as a sequence of R-modules?
   (c) (2 points) Is the sequence split as a sequence of abelian groups?
4. Let R be a commutative ring with unit.
   (a) (2 points) Prove that if I, J are ideals of R then there is an isomorphism

$$R/I \otimes_R R/J \cong R/(I + J).$$

   (b) (2 points) Prove that if R is a PID and M is a finitely generated non-zero R-module, then $M \otimes_R M \neq 0$.
   (c) (1 point) Give an example of a PID R and non-zero R-module N for which $N \otimes_R N = 0$.
5. Let $\mathbb{F}$ be a field of characteristic 2. Let E be the field of fractions of $\mathbb{F}[x]$.
   (a) (1 point) Show that E is an extension of degree 2 of the subfield $\mathbb{F}(x^2 + x)$.
   (b) (2 points) Is E a separable extension of $\mathbb{F}(x^2 + x)$?
   (c) (2 points) Is E a separable extension of $\mathbb{F}(x^2)$?

   Recall that an extension E/F is separable if every element is separable.
6. (5 points) Let $\mathbb{F}$ be a (not necessarily algebraically closed) field, and let $\mathbb{A}^n$ be the n-dimensional affine space of $\mathbb{F}$. Prove that a subset $S \subset \mathbb{A}^n$ is finite if and only if the ring $\mathbb{F}[x_1, \ldots, x_n]/\mathcal{I}(S)$ is finite-dimensional as a vector space over $\mathbb{F}$.