MM7033 – Advanced Algebra

Fall 2023 – Re-exam

- **No** use of textbook, notes, or calculators is allowed.
- Unless told otherwise, you may quote results that were proved in class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.
- In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts

---

1. Let $\mathbb{R}$ be the field of real numbers and consider the affine line $\mathbb{A}^1$ over $\mathbb{R}$.
   (a) (3 points) Let $f(x) = x^4 - 1 \in \mathbb{R}[x]$. Find $\mathcal{Z}(f)$ and $\mathcal{I}\big(\mathcal{Z}(f)\big)$.
   (b) (2 points) Let $\mathbb{N} \subset \mathbb{A}^1$ be the set of non-negative integers. Find $\mathcal{I}(\mathbb{N})$ and $\mathcal{Z}\big(\mathcal{I}(\mathbb{N})\big)$.

2. A field extension E/F is called normal if it is algebraic, and for every $\alpha \in E$ the minimal polynomial of α in $F[x]$ splits completely over E.
   (a) (2 points) Prove that every extension of degree 2 is normal.
   (b) (1 point) Give an example of a degree 3 extension of $\mathbb{Q}$ that is not normal.
   (c) (2 points) Prove that every algebraic extension of a finite field is normal.

3. (5 points) Let B be an abelian group and $A \subset B$ a subgroup. We say that A is a *pure* subgroup if the following holds:

   For all $a \in A$ and all integers $n > 1$, if there exists an element $b \in B$ satisfying $nb = a$ then there exists an element $a_0 \in A$ satisfying $na_0 = a$.

   Prove that A is a pure subgroup of B if and only if for every finitely generated abelian group M, the induced homomorphism $A \otimes M \to B \otimes M$ is injective.

4. Let R be a commutative ring with identity such that $x^2 = x$ for all $x \in R$.
   (a) (2 points) Show that every prime ideal is maximal.
   (b) (3 points) Show that the local ring R_P is a field for every prime ideal P.

5. (5 points) Let $R = \mathbb{F}_2[x]$ and let M be an R-module that satisfies
   (a) $M \simeq (\mathbb{F}_2)^9$ as $\mathbb{F}_2$-modules.
   (b) $M_{x^2+x+1} = R/(x^2) \oplus R/((x-1)^3)$ as R-modules.

   Describe all such R-modules M up to isomorphism. As usual, $M_f := S^{-1}M$ denotes the localization in $S = \{1, f, f^2, \dots\}$.

6. Let R be the ring $\mathbb{Q}[x]/(x^3 - 1)$.
   (a) (3 points) Show that every ideal of R is projective as an R-module.
   (b) (2 points) Show that every R-module is injective. *Hint: By Baer's criterion, to show that an R-module M is injective, it is enough to verify that for every ideal J, the induced homomorphism* $\operatorname{Hom}_R(R, M) \to \operatorname{Hom}_R(J, M)$ *is surjective.*