Department of Mathematics
Stockholm University

# MM7033 – Advanced Algebra

Fall 2024 – Final exam

Examiner: Gregory Arone
Date: January 14, 2025
Time: 14:00–19:00

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- **No** use of textbook, notes, or calculators is allowed.
- Unless told otherwise, you may quote results that were proved in class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.
- In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts.

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1. Let $R$ be a commutative ring.
   (a) (2 points) Recall that an $R$-module $P$ is *projective* if for every surjective homomorphism $M \longrightarrow N$ of $R$-modules, $\mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N)$ is surjective. Prove that a projective module is a direct summand of a free module.
   (b) (3 points) Recall that an $R$-module $P$ is *flat* if for every injective homomorphism $M \longrightarrow N$ of $R$-modules, $M \otimes_R P \longrightarrow N \otimes_R P$ is injective. Prove that every projective module is flat.

2. (5 points) Let $p$ be a prime number and let $F$ be a field of characteristic $p$. Let $E/F$ be a finite field extension. Prove that $F$ is perfect if and only if $E$ is perfect.
   *Suggestion: First show that* $[E^p : F^p] = [E : F]$.

3. (a) (2 points) Give an example of a principal ideal domain $R$ and a torsion-free $R$-module $M$ which is not free.
   (b) (2 points) Give an example of a commutative ring $R$ with identity and a finitely generated torsion-free $R$-module $M$ which is not free.
   (c) (1 point) Is there a principal ideal domain $R$ and a finitely generated torsion-free $R$-module $M$ which is not free?

4. Consider the following algebraic sets over the complex numbers

$$X_1 = \mathcal{Z}(xyz - 1) \subset \mathbb{A}^3, \qquad X_2 = \mathcal{Z}(xy - x^2z, x^2y) \subset \mathbb{A}^3, \qquad X_3 = \mathcal{Z}(x^2 - x, y - z) \subset \mathbb{A}^3.$$

   (a) (3 points) Determine the irreducible components of $X_1$, $X_2$ and $X_3$. Also determine the coordinate ring of every irreducible component.
   (b) (2 points) For each pair of irreducible components (possibly of different $X_i$), determine if they are isomorphic or not.

   *Recall that if $X = V_1 \cup V_2 \cup \cdots \cup V_n$ where the $V_i$ are irreducible, then the $V_i$ are called the irreducible components of $X$.*

*Please turn the page*

5. Let $M_{a,b}$ be the abelian group of $a \times b$-matrices with integer entries. Then matrix multiplication makes $M_{\ell,\ell}$ into a ring and $M_{a,b}$ into a right $M_{b,b}$-module and a left $M_{a,a}$-module.
   (a) (1 point) Prove that the left $M_{\ell,\ell}$-module $M_{\ell,m}$ is isomorphic to $\bigoplus_{j=1}^{m} M_{\ell,1}$ as left $M_{\ell,\ell}$-modules.
   (b) (1 point) Show that the function $M_{k,\ell} \times M_{\ell,m} \longrightarrow M_{k,m}$ given by matrix multiplication is $M_{\ell,\ell}$-balanced.
   (c) (3 points) Prove that the map in (b) induces an isomorphism of abelian groups
   $$M_{k,\ell} \otimes_{M_{\ell,\ell}} M_{\ell,m} \longrightarrow M_{k,m}.$$
   *Suggestion: Begin with considering the case $k = m = 1$.*

6. A morphism $f\colon X \longrightarrow Y$ in a category $\mathcal{C}$ is said to be an *epimorphism* if for every object $Z$ of $\mathcal{C}$ and every pair of morphisms $g_1, g_2\colon Y \longrightarrow Z$ in $\mathcal{C}$ such that $g_1 \circ f = g_2 \circ f$, it holds that $g_1 = g_2$. For example, one can show that in the category of sets, the epimorphisms are exactly the surjective functions.
   (a) (2 points) Let $\mathcal{CR}ing$ be the category with
   - objects: commutative rings $R$ with identity $1_R$, and
   - morphisms: ring homomorphism $f\colon R_1 \longrightarrow R_2$ such that $f(1_{R_1}) = 1_{R_2}$.

   Let $R$ be a commutative ring with identity and let $S \subseteq R$ be a multiplicative set. Show that the localization map $f\colon R \longrightarrow S^{-1}R$ is an epimorphism in $\mathcal{CR}ing$.
   (b) (3 points) Show that an abelian group can have at most one structure as a $\mathbb{Q}$-module.