Solutions to Final exam in MM7033, 2025-01-14, 14:00–19:00

1. (a) Let P be projective and let $q\colon F \twoheadrightarrow P$ be a surjection from a free R-module. Then since P is projective, the identity map $\mathrm{id}_P\colon P \longrightarrow P$ lifts to a map $s\colon P \longrightarrow F$ such that $q \circ s = \mathrm{id}_P$. That is, s is a section of q. It follows that F is the internal direct sum of $P \cong \operatorname{im} s$ and $K = \ker q$.

   (b) Let $F = P \oplus K$ as in (a) and let $M \longrightarrow N$ be injective. Since $F = \bigoplus_I R$ and direct sums commute with tensor products and $M \otimes_R R = M$, we have that $M \otimes_R F \longrightarrow N \otimes_R F$ is the direct sum of the maps $M \longrightarrow N$

$$M \otimes_R F = \bigoplus_I M \longrightarrow \bigoplus_I N = N \otimes_R F$$

   which is injective. Using once more that tensor products commutes with direct sums, we have that $M \otimes_R F = (M \otimes_R P) \oplus (M \otimes_R K)$ and similarly for N and the injective map $M \otimes_R F \longrightarrow N \otimes_R F$ becomes the direct sum of the maps

$$M \otimes_R P \longrightarrow N \otimes_R P$$
$$M \otimes_R K \longrightarrow N \otimes_R K.$$

   In particular, the first map is injective so P is flat.

2. Recall that F is perfect if and only if $F^p = F$. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a basis for E as an F-vector space, then $\alpha_1^p, \alpha_2^p, \ldots, \alpha_n^p$ is a basis for E^p as an F^p-vector space. Indeed, every element in E^p is of the form x^p for $x \in E$. If $x = \sum_i \lambda_i \alpha_i$ with $\lambda_i \in F$, then $x^p = \sum_i \lambda_i^p \alpha_i^p$ so $\alpha_1^p, \alpha_2^p, \ldots, \alpha_n^p$ spans E^p. If $\sum_i \lambda_i^p \alpha_i^p = 0$, then $\sum_i \lambda_i \alpha_i = 0$ so $\lambda_i = 0$. This shows that $\alpha_1^p, \alpha_2^p, \ldots, \alpha_n^p$ is a basis. Thus, $[E : F] = [E^p : F^p]$.

   Now consider $E/F/F^p$ and $E/E^p/F^p$. Since degrees are multiplicative it follows that $[F : F^p] = [E : E^p]$. Thus, F is perfect if and only if E is perfect.

3. (a) Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$. Then M is torsion-free. There are several ways to prove that $\mathbb{Q}$ is not free. For example, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$ whereas $(\bigoplus_I \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = \bigoplus_I \mathbb{Z}/2\mathbb{Z} \neq 0$. Or observe that $\mathbb{Q}$ is divisible: for every $x \in \mathbb{Q}$ and $n \in \mathbb{Z} \smallsetminus \{0\}$, there exists $y \in \mathbb{Q}$ such that $ny = x$. This is clearly not the case for a free $\mathbb{Z}$-module.

   (b) Let $R = k[x, y]$ for some field k and $M = (x, y) \subset R$. Then M is a submodule of the free module R, hence torsion-free. But M is not free. Again this has many proofs. For example, since M is not principal, the rank would have to be at least 2 but there are no injective maps $f\colon R^2 \longrightarrow M$: if $f(e_1) = a$ and $f(e_2) = b$ then $f(be_1 - ae_2) = 0$. Or, note that $M \otimes_R \operatorname{Frac}(R)$ is free of rank 1 so M would have to be principal.

   (c) No, if R is a principal ideal domain and M is a finitely generated torsion-free module, then by the structure theorem $M = R^n$ is free.

4. (a) We claim that X_1 is a variety. To prove this, it is enough to show that $\mathbb{C}[x, y, z]/(xyz - 1)$ is a domain. But $\mathbb{C}[x, y, z]/(xyz - 1) \cong \mathbb{C}[x, y]_{xy} \subseteq \operatorname{Frac}(\mathbb{C}[x, y])$ so is a domain.

   On X_2, we have that $x^2 y = 0$ so either $x = 0$ or $y = 0$. If $x = 0$, then $xy - x^2 z = 0$. If $y = 0$, then $xy - x^2 z = x^2 z = 0$ so either $x = 0$ or $z = 0$. This gives

$$X_2 = \mathcal{Z}(x) \cup \mathcal{Z}(y, z) = X_{21} \cup X_{22}.$$

The coordinate rings are $\mathbb{C}[x,y,z]/(x) \cong \mathbb{C}[y,z]$ and $\mathbb{C}[x,y,z]/(y,z) \cong \mathbb{C}[x]$ which proves that X_{21} and X_{22} are varieties.

On X_3, we have that $x^2 - x = 0$ so either $x = 0$ or $x = 1$. Then also $y = z$. This gives

$$X_3 = \mathcal{Z}(x, y-z) \cup \mathcal{Z}(x-1, y-z) = X_{31} \cup X_{32}.$$

The coordinate rings are $\mathbb{C}[x,y,z]/(x,y-z) \cong \mathbb{C}[z]$ and $\mathbb{C}[x,y,z]/(x-1,y-z) \cong \mathbb{C}[z]$. As these are integral domains, X_{31} and X_{32} are varieties.

(b) We have that $X_{21} \cong \mathbb{A}^2$ and that $X_{22} \cong X_{31} \cong X_{32} \cong \mathbb{A}^1$. Finally, X_1 is not isomorphic to the other varieties because the coordinate ring $\mathbb{C}[X_1] = \mathbb{C}[x,y,z]/(xyz-1)$ is generated by x, y, z and $xyz = 1$. In particular, x, y, z are all units. This cannot be a polynomial ring since the only units in a polynomial ring is $\mathbb{C}^\times$.

(All the calculations are valid over any field.)

5. (a) As an abelian group $M_{\ell,m}$ is $\mathbb{Z}^{\ell\times m}$. If we write $M_{\ell,m}$ as the direct sum of its columns, it is also a direct sum as $M_{\ell,\ell}$-modules since

$$A \begin{bmatrix} v_1 & v_2 & \dots & v_m \end{bmatrix} = \begin{bmatrix} Av_1 & Av_2 & \dots & Av_m \end{bmatrix}$$

for every $A \in M_{\ell,\ell}$.

(b) Let $A \in M_{\ell,\ell}$, let $B, B' \in M_{k,\ell}$ and let $C, C' \in M_{\ell,m}$. Then

$$(B+B')C = BC + B'C$$
$$B(C+C') = BC + BC'$$
$$(BA)C = B(AC)$$

which shows that $M_{k,\ell} \times M_{\ell,m} \longrightarrow M_{k,m}$ is $M_{\ell,\ell}$-balanced.

(c) As in (a), we also have that $M_{k,\ell} = \bigoplus_{i=1}^{k} M_{1,\ell}$ where we have taken the direct sum of the rows. If B_i denotes the ith row of B and C_j denotes the jth column of C, then B_iAC_j is the (i,j)th entry of BAC. This gives

$$M_{k,\ell} \otimes_{M_{\ell,\ell}} M_{\ell,m} \cong \bigoplus_{i,j} \big(M_{1,\ell} \otimes_{M_{\ell,\ell}} M_{\ell,1}\big) \longrightarrow \bigoplus_{i,j} \mathbb{Z} \cong M_{k,m}.$$

It is thus enough to prove that

$$\varphi\colon M_{1,\ell} \otimes_{M_{\ell,\ell}} M_{\ell,1} \longrightarrow M_{1,1} = \mathbb{Z}$$

is an isomorphism. It is surjective because $\varphi(e_i^T \otimes e_i) = 1$.

If $x, y \in \mathbb{Z}^\ell$, then $x^T \otimes y = x^T \otimes Ae_1 = x^T A \otimes e_1 = (x \cdot y)(e_1^T \otimes e_1)$ where $A = \begin{bmatrix} y & 0 & 0 & \dots & 0 \end{bmatrix}$. This shows that every element of $M_{1,\ell} \otimes_{M_{\ell,\ell}} M_{\ell,1}$ is a multiple of $e_1^T \otimes e_1$ so φ is also injective.

6. (a) Let A be a ring and $g_1, g_2\colon S^{-1}R \longrightarrow A$ be ring homomorphisms such that $h := g_1 \circ f = g_2 \circ f\colon R \longrightarrow A$. Then $h(s)$ is invertible for every $s \in S$, since $h(s)g_1(1/s) = 1$, so by the universal property of localization, there is a *unique* map $g\colon S^{-1}R \longrightarrow A$ such that $g \circ f = h$. In particular, $g = g_1 = g_2$.

(b) Let M be an abelian group, that is, a $\mathbb{Z}$-module. There is a canonical ring homomorphism $h\colon \mathbb{Z} \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$ which takes n to multiplication by n. Giving M the structure of a $\mathbb{Q}$-module is the same as giving a ring homomorphism $g\colon \mathbb{Q} \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$ extending h. Note that $\mathrm{End}_{\mathbb{Z}}(M)$ is typically non-commutative but h and g must both land in the center A of $\mathrm{End}_{\mathbb{Z}}(M)$. We thus have $\mathbb{Z} \longrightarrow A$ and the question is whether there can be multiple extensions $\mathbb{Q} \longrightarrow A$. Since $\mathbb{Z} \longrightarrow \mathbb{Q}$ is a localization, it is an epimorphism so there is at most one extension $\mathbb{Q} \longrightarrow A$.