MM7033 – Advanced Algebra

Fall 2024 – Re-exam

- **No** use of textbook, notes, or calculators is allowed.
- Unless told otherwise, you may quote results that were proved in class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.
- In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts.

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1. For any integer $d \geq 1$, let $\zeta_d = e^{2\pi i/d} \in \mathbb{C}$.
   (a) (1 point) For $d = 4$ and $d = 6$, show that $\mathbb{Q}(\zeta_d)$ is a quadratic field, that is, find $D \in \mathbb{Q}$ such that $\mathbb{Q}(\zeta_d) = \mathbb{Q}(\sqrt{D}) \cong \mathbb{Q}[x]/(x^2 - D)$.
   (b) (2 points) Is there a subfield $K \subseteq \mathbb{Q}(\zeta_{12})$ which is not of the form $\mathbb{Q}(\zeta_d)$?

2. Let R be an integral domain, and Q the quotient field of R.
   (a) (2 points) Prove that $Q/R \otimes_R Q/R = 0$.
   (b) (2 points) Prove that if $I \subset R$ is a non-zero ideal then $Q \otimes_R R/I = 0$.
   (c) (1 point) Prove that $Q \otimes_R R \neq 0$.

3. Let R be a commutative ring with 1, and $I \subset R$ an ideal. Recall that if M is an R-module then IM is the submodule of M generated by elements of the form xm, where $x \in I$, $m \in M$. Note that we have inclusions of submodules $M \supset IM \supset I^2M \supset \cdots$. Let $f\colon M \to N$ be a homomorphism of R-modules.
   (a) (3 points) Suppose the composition $M \xrightarrow{f} N \to N/IN$ is surjective, where $N \to N/IN$ is the quotient homomorphism. Prove that the composition $M \xrightarrow{f} N \to N/I^2N$ is surjective as well.
   (b) (2 points) Recall that I is called a nilpotent ideal if $I^k = 0$ for some $k \geq 1$. Suppose I is a nilpotent ideal, and the homomorphism $\bar{f}\colon M/IM \to N/IN$ induced by f is surjective. Prove that the homomorphism $f\colon M \to N$ is surjective as well.

4. Let R be a commutative ring with 1. Recall that an R-algebra is a commutative ring A together with a ring homomorphism $R \to A$. We say that A is finitely generated if there exists $a_1, a_2, \ldots, a_n \in A$ such that A is the smallest R-algebra containing $a_1, a_2, \ldots, a_n$. Equivalently, there is a surjection $R[x_1, x_2, \ldots, x_n] \to A$ of R-algebras.
   (a) (3 points) Let R be a domain. Show that $K := \mathrm{Frac}(R)$ is a finitely generated R-algebra if and only if the natural map $R_f \to K$ is an isomorphism for some $f \in R$.
   (b) (2 points) Let k be a field, let $k[x]$ be the polynomial ring and consider the localization $R = k[x]_{(x)}$. Show that $\mathrm{Frac}(R)$ is a finitely generated R-algebra.

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5. Let $X = \mathcal{Z}(y^3 - x^4, z^3 - x^5, xz - y^2) \subset \mathbb{A}^3$ (we work over the complex numbers).
   (a) (2 points) Show that there is a morphism of algebraic sets $f\colon \mathbb{A}^1 \to X$ that is *bijective*.
   (b) (2 points) Conclude that X is irreducible and that for every non-zero ideal $I \subseteq \mathbb{C}[X]$, the algebraic set $\mathcal{Z}(I)$ is finite.
   (c) (2 points) Show that the morphism $f^*\colon \mathbb{C}[X] \to \mathbb{C}[\mathbb{A}^1]$ is injective and that f is not an isomorphism.

6. Let $\mathbb{F}$ be a field.
   (a) (2 points) Let $p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 \in \mathbb{F}[x]$ be a monic polynomial and let $V = \mathbb{F}[x]/(p(x))$. Then V is a vector space and multiplication by x gives a linear transformation $T\colon V \longrightarrow V$. Using a suitable basis, prove that $p(x)$ is the characteristic polynomial of T.

   Let V be a finite dimensional $\mathbb{F}$-vector space and let $T\colon V \longrightarrow V$ be a linear transformation. This gives V the structure of an $\mathbb{F}[x]$-module where multiplication by x is T.

   (b) (2 points) Let $p_T(x)$ be the characteristic polynomial of T. Prove the Cayley–Hamilton theorem, that is, prove that $p_T(T)$ is the zero transformation on V.
   (c) (2 points) Suppose that $f(x) \in \mathbb{F}[x]$ is a polynomial such that $f(T)$ is the zero transformation on V. Show that every irreducible factor of the characteristic polynomial $p_T(x)$ divides $f(x)$.