Department of Mathematics
Stockholm University

MM 7033 - Advanced algebra

Fall 2025 - final exam

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- **No** use of textbook, notes, or calculators is allowed.
- Unless told otherwise, you may quote results that are proved in the textbook or were covered in class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.

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1. (a) (3 points) Show that $\frac{\sqrt{3}}{2}$ is algebraic, but not integral over $\mathbb{Z}$.

   (b) (2 points) Is the number $\sqrt{3}+\sqrt[3]{2}$ integral over $\mathbb{Z}$?

2. Let $A=\mathbb{C}[x]$ denote the polynomial ring in the variable x over the complex numbers. The matrix
$$\begin{pmatrix} x^2-1 & (x-1)(x^2+2) \\ x^2+2 & (x+1)(x^2+2) \end{pmatrix}$$
defines an A-module map $A^2 \to A^2$ with cokernel N.

   (a) (3 points) Determine the invariant factors of N.

   (b) (2 points) Determine the maximal ideals $\mathfrak{m} \subseteq A$ such that $N \otimes_A A/\mathfrak{m}$ is non-zero.

3. Let k be a field, and let V be a k-vector space with basis $\{x, y, z, w\}$.

   (a) (3 points) Express $(x+2y+3z+4w)\wedge(2y+z)\wedge(y+w)$ in a basis of $\wedge^3 V$.

   (b) (2 points) [Added during exam: Assume that k is a field of characteristic 2]. We have the k-linear map $S_2(V) \to \wedge^2 V$ from the the second symmetric power to the second exterior power of V, sending $v \otimes u \mapsto v \wedge u$. Give a basis of the kernel of this map.

4. Let $\mathbb{Z}[x]$ denote the ring of polynomials in the variable x with integer coefficients. Describe the following tensor products as explicitly as possible:

   (a) (2 points) $\mathbb{Z}[x]/(2) \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x+4)$.

   (b) (3 points) $\mathbb{Z}[x]/(2) \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x]/(x+4)$.

5. (a) (2 points) Define the notion of an injective module.

   (b) (3 points) Show that $\mathbb{Z}/4$ is an injective module over itself, but is not an injective module over $\mathbb{Z}$.

6. Let P be an A-module, and let C denote the category of A-modules. We have the functor $\mathrm{Hom}_A(P,-)\colon C \to \mathrm{Ab}$. Prove or disprove that this functor is exact, when

   (a) (2 points) we have $A=\mathbb{Z}$, and $P=\mathbb{Z}/(3)$,

   (b) (3 points) we have $A=\mathbb{Z}/(6)$, and $P=\mathbb{Z}/(3)$.