Department of Mathematics
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MM 7033 - Advanced algebra

Fall 2025 - final exam

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- **No** use of textbook, notes, or calculators is allowed.
- Unless told otherwise, you may quote results that are proved in the textbook or were covered in class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.

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1. (a) (3 points) Show that $\frac{\sqrt{3}}{2}$ is algebraic, but not integral over $\mathbb{Z}$.

   Solution: $\frac{\sqrt{3}}{2}$ is algebraic over $\mathbb{Z}$ because it is a root of the polynomial $4x^2 - 3 \in \mathbb{Z}[x]$. There are many ways to show that it is not integral, here is one: if $\frac{\sqrt{3}}{2}$ is integral, then so is $\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$. But we know that non-integer rationals are never integral over $\mathbb{Z}$.

   (b) (2 points) Is the number $\sqrt{3} + \sqrt[3]{2}$ integral over $\mathbb{Z}$?

   Solution: Yes. It is easy to see that $\sqrt{3}$ and $\sqrt[3]{2}$ are both integral, and the sum of two integral numbers is again integral.

2. Let $A = \mathbb{C}[x]$ denote the polynomial ring in the variable x over the complex numbers. The matrix

$$\begin{pmatrix} x^2 - 1 & (x-1)(x^2+2) \\ x^2 + 2 & (x+1)(x^2+2) \end{pmatrix}$$

   defines an A-module map $A^2 \to A^2$ with cokernel N.

   (a) (3 points) Determine the invariant factors of N.

   Solution: We subtract the second row from the first, and get that the second row then becomes $\begin{pmatrix} 3 & 2(x^2+2) \end{pmatrix}$. We thereafter add $\frac{1}{3}(1 - x^3)$ of this new second row, to the first. The matrix is now

$$\begin{pmatrix} 0 & (x^2+2)(\frac{2}{3} - \frac{2}{3}x^2 + x - 1) \\ 3 & 2(x^2+2) \end{pmatrix} = \begin{pmatrix} 0 & (x^2+2)2(x^2 - \frac{3}{2}x + \frac{1}{2}) \\ 3 & 2(x^2+2) \end{pmatrix}.$$

   After the obvious elementary column operations we see that the invariant factors are (1) and $(x^2+2)(x - \frac{1}{2})(x-1)$.

   (b) (2 points) Determine the maximal ideals $\mathfrak{m} \subseteq A$ such that $N \otimes_A A/\mathfrak{m}$ is non-zero.

   Solution: Independently from a) we compute the determinant of the matrix as

$$(x-1)(x^2+2)\Big((x+1)^2 - (x^2+2)\Big) = (x-1)(x^2+2)(2x-1).$$

   The irreducible factors are then $(x-1)$, $(x \pm i\sqrt{2})$ and $(2x-1)$, giving us four different maximal ideals in A. For all other maximal ideals $\mathfrak{m} \subset A$ we have that the determinant in non-zero in $A/\mathfrak{m}$, hence the map is invertible and consequently $N \otimes_A A/\mathfrak{m} = 0$. For the singled out four maximal ideals, however, the determinant is zero in $A/\mathfrak{m}$, so the map is not surjective, and consequently $N \otimes_A A/\mathfrak{m} \neq 0$.

3. Let k be a field, and let V be a k-vector space with basis $\{x, y, z, w\}$.

(a) (3 points) Express $(x + 2y + 3z + 4w) \wedge (2y + z) \wedge (y + w)$ in a basis of $\wedge^3 V$.

Solution: Multiplying out, and using that $u \wedge u = 0$ for every vector u, we find that

$$(x+2y+3z+4w)\wedge(2y+z)\wedge(y+w) = 2x\wedge y\wedge w+x\wedge z\wedge y+x\wedge z\wedge w+2y\wedge z\wedge w+6z\wedge y\wedge w+4w\wedge z\wedge y.$$

Using the skew-symmetry of exterior product, the answer simplifies to

$$-x \wedge y \wedge z + 2x \wedge y \wedge w + x \wedge z \wedge w - 8y \wedge z \wedge w.$$

Since $x \wedge y \wedge z, x \wedge y \wedge w, x \wedge z \wedge w$ and $y \wedge z \wedge w$ form a basis for $\wedge^3 V$, this is an answer.

(b) (2 points) [Added during exam: Assume that k is a field of characteristic 2]. We have the k-linear map $S_2(V) \to \wedge^2 V$ from the the second symmetric power to the second exterior power of V, sending $v \otimes u \mapsto v \wedge u$. Give a basis of the kernel of this map.

Solution: We definitely have elements of the form $v \otimes v$ in the kernel. The map is surjective, and co-domain has dimension $\binom{4}{2} = 6$, and the symmetric tensor algebra in degree 2 has dimension $\binom{4+1}{2} = 10$. So the kernel is of dimension 4. So, a basis of the kernel are the four squares $x \otimes x, y \otimes y, z \otimes z$ and $w \otimes w$.

4. Let $\mathbb{Z}[x]$ denote the ring of polynomials in the variable x with integer coefficients. Describe the following tensor products as explicitly as possible:

(a) (2 points) $\mathbb{Z}[x]/(2) \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x + 4)$.

Solution: We have that there are isomorphisms of $\mathbb{Z}$-modules $\mathbb{Z}[x]/(2) \cong \mathbb{Z}/(2)[x]$ and $\mathbb{Z}[x]/(x+4) \cong \mathbb{Z}$. Therefore the sought tensor product is $\mathbb{Z}/(2)[x] \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}/(2)[x]$, this is the polynomial ring with coefficients in $\mathbb{Z}/(2)$.

(b) (3 points) $\mathbb{Z}[x]/(2) \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x]/(x + 4)$.

Solution: We have that the tensor product $\mathbb{Z}[x]/(2)\otimes_{\mathbb{Z}[x]}\mathbb{Z}[x]/(x+4)$ equals $\mathbb{Z}[x]/(2, x+4)$. The ideal $(2, x + 4)$ equals $(2, x)$. And we have that the quotient $\mathbb{Z}[x]/(2, x) = \mathbb{Z}/(2)$ is the field with two elements.

5. (a) (2 points) Define the notion of an injective module.

(b) (3 points) Show that $\mathbb{Z}/4$ is an injective module over itself, but is not an injective module over $\mathbb{Z}$.

Solution: As $\mathbb{Z}$ is a PID and the multiplication map $x \mapsto 2x$ on $M = \mathbb{Z}/(4)$ is not surjective, we have that M is not divisible, and therefore not an injective $\mathbb{Z}$-module.

To see that M is injective as a module over itself $A = \mathbb{Z}/(4)$, we use the Baer criterion. So $\mathbb{Z}/(4)$ is injective as an $\mathbb{Z}/(4)$-module if for any ideal I of $\mathbb{Z}/(4)$, any morphism $I \to \mathbb{Z}$ can be extended to a morphism $\mathbb{Z}/(4) \to \mathbb{Z}/(4)$. There are few ideals in $\mathbb{Z}/(4)$, these are $(0), (2)$ and (1). Maps from (0)-ideals are uninteresting, and maps from $(1) = \mathbb{Z}/(4)$ impose no conditions. So, we are left considering maps $(2) \to \mathbb{Z}/(4)$. There are potentially four maps, each determined by sending 2 to any of the four elements in $\mathbb{Z}/(4)$. The one sending 2 to 0 is the zero map, and extends. The one sending 2 to 2 extends to the identity map on $\mathbb{Z}/(4)$. The two remaining maps, sending 2 to 1, or sending 2 to 3, will not be $\mathbb{Z}/(4)$-linear. Indeed, if the map $f\colon (2) \to \mathbb{Z}/(4)$ is such that $f(2) = 1$ (or $f(2) = 3$), then we have $2 \cdot f(2) = 2$ (or $2 \cdot f(2) = 2$), but also that $2 \cdot f(2) = f(4) = f(0) = 0$. It follows that $\mathbb{Z}/(4)$ is injective as a module over itself.

6. Let P be an A-module, and let C denote the category of A-modules. We have the functor $\mathrm{Hom}_A(P, -)\colon C \to \mathrm{Ab}$. Prove or disprove that this functor is exact, when

(a) (2 points) we have $A = \mathbb{Z}$, and $P = \mathbb{Z}/(3)$,

Solution: We know that the functor $\mathrm{Hom}_A(P, -)$ is left exact, and moreover that it is exact if and only if P is a projective module.

We want to show that $P = \mathbb{Z}/(3)$ is not a projective $\mathbb{Z}$-module. This is equivalent with P not being a summand of a free-module. It is clear that a summand of a free $\mathbb{Z}$-module is torsion-free, while $\mathbb{Z}/(3)$ is not torsion-free, so it is not projective.

One can also see that $\mathbb{Z}/(3)$ is not projective by applying $\hom(\mathbb{Z}/(3), -)$ to the surjective homomorphism $\mathbb{Z} \to \mathbb{Z}/(3)$. It is easy to see that the resulting homomorphsim $\hom(\mathbb{Z}/(3), \mathbb{Z}) \to \hom(\mathbb{Z}/(3), \mathbb{Z}/(3))$ is isomorphic to the non-surjective homomorphism $0 \to \mathbb{Z}/(3)$.

(b) (3 points) we have $A = \mathbb{Z}/(6)$, and $P = \mathbb{Z}/(3)$.

Solution: Here we claim that $P = \mathbb{Z}/(3)$ is a direct summand of a free $\mathbb{Z}/(6)$-module. Indeed, by the Chinese Remainder Theorem there is an isomorphism $\mathbb{Z}/(6) \cong \mathbb{Z}/(3) \times \mathbb{Z}/(2)$. This is an isomorphism of rings, and in particular of $\mathbb{Z}$-modules, and therefore also of $\mathbb{Z}/(6)$-modules. So $\mathbb{Z}/(3)$ is a direct summand of $\mathbb{Z}/(6)$.