FINAL EXAM

Instructions: Justify your answers. You may use results from the homework sets, but make sure to carefully state such results. No calculators and no notes allowed.

Grading: This exam is worth 30 points. If you completed homework assignments, your homework bonus (out of 3 points) will be added to your score. You need a score of 12.5/30 or higher to pass this exam. More precisely, the following scale will be used:

A: $[26.5, 30]$, B: $[23, 26.5)$, C: $[19.5, 23)$, D: $[16, 19.5)$, E: $[12.5, 16)$, F: $[0, 12.5)$.

Problem 1. *Let $f(x) = x^{13} - 15 \in \mathbf{Q}[x]$.*

(a) (1 point) Show that f is irreducible over $\mathbf{Q}$.
(b) (2 points) Give an explicit description of a splitting field L for f over $\mathbf{Q}$.
(c) (1 point) Compute $[L : \mathbf{Q}]$. Justify your answer.
(d) (1 point) Show that $L/\mathbf{Q}$ is Galois.

Problem 2. *Let f and L be as in Problem 1.*

(a) (2 points) Give generators and relations for $\mathrm{Gal}(L/\mathbf{Q})$.
(b) (2 points) Show that $\mathrm{Gal}(L/\mathbf{Q})$ is solvable.
(c) (2 points) Show that there is a unique extension $K/\mathbf{Q}$ of degree 12 which is contained in L.
(d) (2 points) Show that there is a unique quadratic extension $F/\mathbf{Q}$ contained in L and describe F as $\mathbf{Q}(\sqrt{D})$ for some integer D.

Problem 3. *Let $\Phi_{24}(x) \in \mathbf{Z}[x]$ be the cyclotomic polynomial of primitive 24th roots of unity. Let ζ be a root of $\Phi_{24}(x)$ in some finite extension of $\mathbf{Q}$.*

(a) (2 points) Show that for every prime p, the reduction of $\Phi_{24}(x)$ modulo p is reducible in $\mathbf{F}_p[x]$.
(b) (2 points) Is the regular 24-gon constructible by straightedge and compass? Justify your answer.
(c) (2 points) Show that there are precisely 7 quadratic extensions of $\mathbf{Q}$ contained in $\mathbf{Q}(\zeta)$.

Problem 4. *Let $f(x) = x^4 + ax^2 + b \in \mathbf{Q}[x]$.*

(a) (2 points) Show that the roots of f in a splitting field have the form $\pm\alpha, \pm\beta$ and that $(\alpha\beta)^2 \in \mathbf{Q}$.
(b) (2 points) Show that $f(x)$ is irreducible over $\mathbf{Q}$ if and only if none of $\alpha^2, \alpha+\beta$ and $\alpha-\beta$ lie in $\mathbf{Q}$.
(c) (2 points) Assume f is irreducible. Show that the Galois group of f has order 4 or 8.
(d) (2 points) Assume f is irreducible. Show that the Galois group of f is the Klein 4-group $\mathbf{Z}/2 \times \mathbf{Z}/2$ if and only if $\alpha\beta \in \mathbf{Q}$.

Problem 5.

(a) (1 point) Show that $x^3 - 2$ divides $x^{343} - x$ in $\mathbf{F}_7[x]$.
(b) (2 points) Show that the 8th cyclotomic polynomial $\Phi_8(x) = x^4 + 1$ divides $x^{p^2} - x$ in $\mathbf{F}_p[x]$ for every odd prime p.

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