FINAL EXAM

Must be submitted, preferably by the course website, else by email, at the latest by:
16:00 on 2020-12-18 (unless you have been granted extra time)

1. Instructions

Justify your answers. Since this is an exceptional "zoom-pandemic-exam" you may use notes, homework and texts associated with the course (tablet notes on the course website, the text by Dummit & Foote), but you should not search on the internet for answers. You may e.g., use part of Problem 4 to do part of Problem 1, even if you are unsuccessful with that part of Problem 4. You may use part (a) of a problem to do part (b) even if you have not solved (a), and so on. You can say "By Homework 3, problem 2,...". You do not need to restate the question in your solution. **Please email me if you have any questions during the exam**.

Grading: This exam is worth 30 points. If you completed homework assignments, your homework bonus (out of 3 points) will be added to your score. You need a score of 12.5/30 or higher to pass this exam. More precisely, the following scale will be used:

A: $[26.5, 30]$, B: $[23, 26.5)$, C: $[19.5, 23)$, D: $[16, 19.5)$, E: $[12.5, 16)$, F: $[0, 12.5)$.

2. Problems

There are 5 problems:

Problem 1 (5 points)**.** *Let $f(x) = x^{11} - 29 \in \mathbf{Q}[x]$.*

(a) *(1 point) Show that f is irreducible over $\mathbf{Q}$.*
(b) *(2 points) Give an explicit description of a splitting field L for f over $\mathbf{Q}$.*
(c) *(1 point) Compute $[L : \mathbf{Q}]$. Justify your answer.*
(d) *(1 point) Show that $L/\mathbf{Q}$ is Galois.*

Problem 2 (7 points)**.** *Let f and L be as in Problem 1.*

(a) *(2 points) Give generators and relations for* $\mathrm{Gal}(L/\mathbf{Q})$.
(b) *(2 points) Show that* $\mathrm{Gal}(L/\mathbf{Q})$ *is solvable.*
(c) *(3 points) Show that there is a unique quadratic extension $F/\mathbf{Q}$ contained in L and describe F as $\mathbf{Q}(\sqrt{D})$ for some integer D.*

Problem 3 (6 points)**.** *On this problem, if you solve (b) you can cite it in (a). But (a) can also be done by different methods which you may find easier (so you might be able to do (a) even if you don't manage (b)).*

(a) *(1 point) Show that a subgroup of S_5 containing a* 5*-cycle and a transposition is all of S_5.*
(b) *(2 point)* Let p be a prime. *Show that a subgroup of S_p containing a p-cycle and a transposition is all of S_p.*
(c) *(1 point) Prove or give a counterexample to the following statement: "For every integer $n \geq 2$, if a subgroup H of S_n contains an n-cycle and a transposition, then $H = S_n$".*
(d) *(2 point) Assume that $f \in \mathbf{Q}[x]$ is irreducible of degree* 5*, that f is solvable by radicals and that the discriminant of f is negative. What is the order of* $\mathsf{Gal}(f)$*?*

Problem 4 (4 points)**.**

(a) *(1 points) Let $f(x) = x^n - x + b \in \mathbf{Z}[x]$ and let q be a prime divisor of b. Show that f is separable in $\mathbf{F}_q[x]$ if and only if q does not divide $n - 1$.*
(b) *(3 points) Show that the Galois group $\mathsf{Gal}(f) \subset S_{13}$ of $f(x) = x^{13} - x + 385$ over $\mathbf{Q}$ contains a* 13*-cycle, as well as elements of cycle type*

$$(2, 2, 2),\ (2, 2, 2, 2),\ \text{and}\ (2, 2, 2, 2, 2)$$

(here e.g., $(2, 2, 2)$ means a product of three disjoint transpositions).

Problem 5 (8 points)**.** *Let ζ_n be a primitive nth root of unity in some extension of $\mathbf{Q}$.*

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(a) *(2 points) Find* $m_{\zeta_7+\zeta_7^{-1},\mathbf{Q}}(x)$.

(b) *(1 point) Is* $\cos(2\pi/7)+5$ *constructible by straightedge and compass?*

(c) *(2 points) Let* p *be a prime different from* 7. *Show that* $m_{\zeta_7+\zeta_7^{-1},\mathbf{Q}}(x)$ *is irreducible in* $\mathbf{F}_p[x]$ *if* $p \not\equiv \pm 1 \pmod 7$ *and that otherwise* $m_{\zeta_7+\zeta_7^{-1},\mathbf{Q}}(x)$ *splits completely in* $\mathbf{F}_p[x]$.

(d) *(2 points) Recall that* 97 *is prime. Let* $\mathbf{F}_{97}^{\times,3}$ *be the subgroup of cubes in* $\mathbf{F}_{97}^{\times}$. *Define*

$$\alpha = \sum_{a \in \mathbf{F}_{97}^{\times,3}} \zeta_{97}^{a}.$$

Determine the degree of α *over* $\mathbf{Q}$.

(e) *(1 points) Let* α *be the element defined in (d). Is* α *solvable by radicals? Justify your answer.*