MATEMATISKA INSTITUTIONEN	Tentamensskrivning i
STOCKHOLMS UNIVERSITET	MM7043
Avd. Matematik	2025-01-11
Examinator: Sofia Tirabassi	

Instructions:

- During the exam you MAY NOT use textbooks, class notes, or any other supporting material.
- To solve a bullet point in the given exercise you can use all the preceeding points, even if you have not provided a solution for them.
- Start every problem on a new page, and write at the top of the page which problem it belongs to. (But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear and wrong argument, even if the final answer is correct.
- Write clearly and legibly.

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty

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(1) **Field extensions** Consider the polynomial $f(x) = x^{13} - 10 \in \mathbb{Z}[x]$.
  (a) (1 pt) Show that $f(x)$ is irreducible over $\mathbb{Q}$.
  (b) (2 pts) Give an explicit description of L, the splitting field of $f(x)$ over $\mathbb{Q}$, as a subfield of $\mathbb{C}$.
  (c) (1 pt) Compute $[L : \mathbb{Q}]$. Justify your answer.
  (d) (1 pt) Show that $L/\mathbb{Q}$ is Galois.

(2) **Generators and relations** Let $f(x)$ and L be as in the previous problem:
  (a) (2 pts) Show that $\mathrm{Gal}(L/\mathbf{Q})$ has a unique normal subgroup N of index 12. Describe explicitly the fixed field L^N.
  (b) (2 pts) Give generators and relations for $\mathrm{Gal}(L/\mathbf{Q})$.
  (c) (2 pts) Show that $\mathrm{Gal}(L/\mathbf{Q})$ is solvable.

(3) **Cyclotomic extensions** Let $\Phi_{36}(x) \in \mathbb{Z}[x]$ denote cyclotomic polynomial of primitive 36 roots of unity and let ξ a root of $\Phi_{36}(x)$ in some field extension of $\mathbb{Q}$.
  (a) (2 pts) Show that $\mathbb{Q}(\xi)/\mathbb{Q}$ is Galois and compute $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$. Is $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ isomorphic to a cyclic group?
  (b) (2 pts) Show that $\Phi_{36}(x) \in \mathbb{F}_p[x]$ is reducible for every prime p.
  (c) (2 pts) Determine wether the regular 36-gon is constructible with straight edge and compass.
  (d) (3 pts) Determine how many quadratic extensions of $\mathbb{Q}$ are contained in $\mathbb{Q}(\xi)$, and give them explicitly as $\mathbb{Q}(\sqrt{D})$ for an integer D.

(4) **Galois group of polynomials** Consider the $g(x) = x^4 + 4x^3 + 4x^2 + x - 2$ in $\mathbb{Z}[x]$.
  (a) (2 pt) Let $g_2(x)$ and $g_3(x)$ denote the reductions of $g(x)$ mod 2 and mod 3, respectively. Show that they are separable.
  (b) (2 pts) Show that $\mathrm{Gal}(g_3)$ is cyclic of order k such that $k|4$.
  (c) (3 pts) Show that $g_3(x)$ is irreducible over $\mathbb{F}_3[x]$.

(d) (2 pt) Determine the image of $\mathrm{Gal}(g)$ in S_4. You can use, without proof, that the transitive subgroups of S_4 are

$$V_4, \quad C_4, \quad D_8, \quad A_4, \quad \text{and } S_4.$$

(e) (1 pt) Assume that $g(x)$ admits a real root (in fact it admits 2 of them). Decide whether this is constructible with straight edge and compass.

GOOD LUCK!!!