MATEMATISKA INSTITUTIONEN	Tentamensskrivning i
STOCKHOLMS UNIVERSITET	MM7043
Avd. Matematik	2025-02-11
Examinator: Sofia Tirabassi	

Instructions:

- During the exam you MAY NOT use textbooks, class notes, or any other supporting material.
- To solve a bullet point in the given exercise, you can use all the preceding points, even if you have not provided a solution for them.
- Start every problem on a new page and write at the top of the page to which problem it belongs.(But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for an unclear or wrong argument, even if the final answer is correct.
- Write clearly and legibly.

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(1) **Field extensions** Consider the polynomial $f(x) = x^{11} - 98 \in \mathbb{Z}[x]$.
  (a) (1 pt) Show that $f(x)$ is irreducible over $\mathbb{Q}$.
  (b) (2 pts) Give an explicit description of L, the splitting field of $f(x)$ over $\mathbb{Q}$, as a subfield of $\mathbb{C}$.
  (c) (1 pt) Compute $[L : \mathbb{Q}]$. Justify your answer.
  (d) (1 pt) Show that $L/\mathbb{Q}$ is Galois.

(2) **Generators and relations** Let $f(x)$ and L be as in the previous problem:
  (a) (2 pts) Show that $\mathrm{Gal}(L/\mathbf{Q})$ has a unique normal subgroup N of index 10. Describe explicitly the fixed field L^N.
  (b) (2 pts) Give generators and relations for $\mathrm{Gal}(L/\mathbf{Q})$.
  (c) (2 pts) Show that $\mathrm{Gal}(L/\mathbf{Q})$ is solvable.

(3) **Cyclotomic extensions** $\Phi_{10}(x) \in \mathbb{Z}[x]$ denote cyclotomic polynomial of primitive 10-th roots of unity and let ξ a root of $\Phi_{10}(x)$ in some field extension of $\mathbb{Q}$.
  (a) (2 pts) Show that $\mathbb{Q}(\xi)/\mathbb{Q}$ is Galois and compute $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$. Is $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ isomorphic to a cyclic group?
  (b) (2 pts) Let $p \neq 2, 5$ be a prime. Show that $\Phi_{10}(x) \in \mathbb{F}_p[x]$ is irreducible in $\mathbb{F}_p$ if and only if $p \equiv 3$ or $p \equiv 7 \mod 10$.
  (c) (1 pts) Show that $\Phi_{10}(x)$ is reducible modulo 5 but irreducible modulo 2.
  (d) (2 pts) Determine whether the regular 10-gon is constructible with straight edge and compass.
  (e) (2 pts) Show that $\mathbb{Q}(\xi)$ contains a unique subextension that is quadratic over $\mathbb{Q}$ and give it explicitly as $\mathbb{Q}(\sqrt{D})$ for an integer D.

(4) **Geometric constructions** (5 pts) Let $f(x)$ an irreducible polynomial of degree 4 with at least a real root. Then the Galois group can be isomorphic to any of the following subgroup of S_4:

$$V_4, \quad C_4, \quad D_8, \quad A_4, \quad \text{and } S_4.$$

Determine in each of the 5 cases above whether a real root of α can be constructed with a straight edge and a compass.

(5) **Galois group of polynomials** Let p be a prime and consider the polynomial $f(x) = x^4 + px + p$ in $\mathbb{Q}[x]$

(a) (2 pts) Recall that the resolvent of $f(x)$ is given by $r(x) = x^3 - 4px - p^2$. Show that $r(x)$ is irreducible when $p \neq 3, 5$

(b) (2 pts) Show that the polynomial $x^4 + x + 1$ is separable and does not divide $x^4 - x$ over $\mathbb{F}_2[x]$

(c) (1 pt) Deduce that $\mathrm{Gal}(f) \simeq S_4$ when p is odd and $\neq 3, 5$.

GOOD LUCK!!!