Galois Theory Exam February 11 2025

Exercise 1

(a) Eisenstein with $p=2$

(b) let ξ a primitive 11-root of 1

$$L = \mathbb{Q}(\sqrt[11]{98}, \xi)$$

The roots of $f(x)$ are $\xi^i \sqrt[11]{98}$ $i=0 \dots 10$

Thus the polynomial splits in L. Suppose that the polynomial splits in F then

$$\sqrt[11]{98} \in F \quad \text{and}$$

$$\xi = \frac{\xi \sqrt[11]{98}}{\xi} \in F \implies F \supseteq L$$

proving that L is the splitting field

(c) $[L:\mathbb{Q}] = \underset{\underset{11}{\wedge\!\!\!\backslash}}{[L:\mathbb{Q}(\xi)]}\ \underset{\underset{10}{\shortparallel}}{[\mathbb{Q}(\xi)\ \mathbb{Q}]} \leq 11 \cdot 10$

since ~~Ber~~ $\sqrt[11]{98}$ is a root of a deg 11 poly on $\mathbb{Q}(\xi)[x]$

but both $[\mathbb{Q}(\xi):\mathbb{Q}]$ & $[L:\mathbb{Q}]$

$[\mathbb{Q}(\sqrt[11]{98}):\mathbb{Q}] \mid [L:\mathbb{Q}]$

$$\implies [L:\mathbb{Q}] \geq \operatorname{lcm}(11,10) = 11 \cdot 10$$

we conclude $[L:\mathbb{Q}] = 110$.

(d) L is the splitting field of f which is irreducible & hence separable since char $\mathbb{Q}=0$

$\implies L$ is Galois

Exercise 2

(a) let $G = Gal(f)$ $\qquad |G| = 110$

n_{11} = number of 11-sylow group

$n_{11} \equiv 1 \mod 11$ and $n_{11} | 10$

$\Rightarrow n_{11} = 1$

There is a unique subgroup of G of order 11 (and index 10) which is normal by Sylow theorem.

$L^N / \mathbb{Q}$ is going to be Galois with $[L^N : \mathbb{Q}] = 10$. Thus $L^N = \mathbb{Q}(\zeta)$

In fact $\mathbb{Q}(\zeta)/\mathbb{Q}$ is Galois $[\mathbb{Q}(\zeta):\mathbb{Q}] = 10$ by the correspondence

$Gal(L/\mathbb{Q}(\zeta)) = N$ since N unique of index 10

Thus $\mathbb{Q}(\zeta)' = L^N$.

(b) ζ^2 generates μ_{11}

Thus we have two generato

$\tau(\zeta) = \zeta^2 \qquad \tau(\sqrt[11]{98}) = \sqrt[11]{98}$

$\sigma(\zeta) = \zeta \qquad \sigma(\sqrt[11]{98}) = \zeta\sqrt[11]{98} \qquad N = \langle\sigma\rangle$

For the relations we have just hav $\langle\tau\rangle$ acts by conjugation on σ

$\tau\sigma\tau^{-1}(\zeta) = \zeta$

$\tau\sigma\tau^{-1}(\sqrt[11]{98}) = \tau(\zeta\sqrt[11]{98}) = \zeta^2\sqrt[11]{98} = \sigma^2(\sqrt[11]{98})$

thus

$$G = \langle \sigma, \tau \mid \sigma^{11} = \tau^{10} = 1 \quad \tau\sigma = \sigma^2\tau \rangle$$

(c) The extension is clearly radical, so G is solvable. More accurately

$$\cancel{1} \lhd N \lhd G$$

G/N is a ~~[illegible]~~ cyclic group generated by $\langle \tau N$.

Exercise 3

(a) $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \simeq (\mathbb{Z}/10)^* \simeq (\mathbb{Z}/2)^* \times (\mathbb{Z}/5)^*$

$\simeq \mathbb{Z}/4$

(b) $\deg \Phi_{10} = 4$

we know that if $p \neq 2, 5$ then

$p \bmod 10 \in (\mathbb{Z}/10)^* = \{1, 3, 7, 9\}$

if $p \equiv 1, 9 \bmod 10$ then we have that

$p^2 \equiv 1 \bmod 10$

$10 \mid p^2 - 1$

$x^{10} - 1 \mid x^{p^2-1} - 1 \mid x^{p^2} - x$

$x^{10} - 1$ has irreducible factor at most of degree 2.

If $p \equiv 3, 7 \bmod 10$ then p has order 4 in $(\mathbb{Z}/10)^*$

This means that

$x^{10} - 1 \mid x^{p^4} - x$

but $x^{10} - 1$ does not divide any

$x^{p^k} - x$ for $k < 4$

$\Rightarrow x^{10} - 1$ has factors of degree 4 and

thus Φ_{10} is irreducible

(c) for $p=5$

$x^{10}-1 = (x^2-1)^5$ has only factor of degree 2

so Φ_{10} is irreducible.

for $p=1$ $\quad \Phi_5 \equiv \Phi_{10} \mod 2$

Φ_5 (and 2 has order 4 in $(\mathbb{Z}/5)^\times$

$\Rightarrow \Phi_{10}(x)$ is irreducible

(d) $10 = 2\cdot 5 = 2\cdot(2^2+1)$ is a product of a power of 2 and an F- prime

YES, it is constructible

(e) $(\mathbb{Z}/10)^* \simeq \mathbb{Z}/4$ which has a unique subgroup of index 2.

Thus there is only one quadratic extension / $\mathbb{Q}$

This is given by $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}(\mu_5) \subseteq \mathbb{Q}(\zeta)$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{\mathbb{Q}(\zeta+\zeta^9)}$

Exercise 4

(a) Denote by α a real root of $f(x)$

V_4 has a subgroup of index 2 which correspond to an extension

$L \subseteq Spl(f(x))$

$[L : \mathbb{Q}] = 2$ CONSTRUCTIBLE

Note that in this case

$Spl(f(x)) = \mathbb{Q}(\alpha)$

since $[Spl(f(x)) : \mathbb{Q}] = 4$

C_4. Same as before $Spl(f(x)) = \mathbb{Q}(\alpha)$

$Spl(f(x))$ has a subextension L quadratic $/ \mathbb{Q}$.

CONSTRUCTIBLE

D_8 See solution exam 21/01/2025

CONSTRUCTIBLE

A_4 NOT CONSTRUCTIBLE

S_4 Done in class

Not constructible

Exercise 5

(a) it is a polynomial of degree 3 so it is enough to check that it has no roots.

possible roots $\pm p \quad \pm p^2$

$$r(p) = p^3 - 4p^2 - p^2 = p^3 - 5p^2 = p^2(p-5)$$
$$= 0 \quad \text{iff} \quad p = 5$$

$$r(-p) = -p^3 + 4p^2 - p^2 = -p^3 + 3p^2 = p^2(3-p)$$
$$= 0 \quad \text{iff} \quad p = 3$$

$$r(p^2) = p^6 - 4p^3 - p^2 = p^2 \underbrace{(p^4 - 4p + 1)}_{\text{cannot be } 0 \text{ since it is not divisible by } p} \neq 0$$

$$r(-p^2) = -p^6 + 4p^3 - p^2 = p^2 \overbrace{(-p^4 + 4p - 1)} \neq 0$$

Thus the polynomial has a root iff ~~p~~ $p = 3, 5$

(b) $f'(x) = 1$ thus $\gcd(f(x), f'(x)) = 1$ and f is separable

It is irreducible since it has no root and $\neq (x^2+x+1)^2$ which is the only way a poly of deg 4 can be written as product of two irred of deg 2 in $\mathbb{F}_2[x]$

(c) $f(x)$ is irreducible over $\mathbb{F}_2[x] \Rightarrow$
Gal(f) contains a 4-cycle
$\mathrm{Gal}(f) \not\leq A_4$

Since the resolvent is irreducible we have that

$$\mathrm{Gal}(f) = S_4, \ A_4$$

but we just excluded the latter.