Department of Mathematics, Stockholm University

Exam in MT5017, Theory of Statistical Inference, December 4, 2025, 14:00–19:00.

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Allowed aids: None.

Return: Communicated via course forum.

Arguments and computations should be clear and easy to follow.

- - - - - - - - PART 1 - - - - - - - -

Problem 1

Let $x_1, \dots, x_n$ be realizations of $X_1, \dots, X_n$ which are independent copies of X which is uniformly distributed on the interval $[0, \theta]$, $\theta > 0$.

(a) Determine the MLE of θ. (5 p)

(b) Determine a one-dimensional sufficient statistic $h(X_1, \dots, X_n)$ for θ. (10 p)

Problem 2

Population data reveal that 1% of the population has a particular decease. Consider a medical test with the property that the test is negative with probability 90% for an individual who does not have the decease, and the test is positive with probability 90% for an individual who has the decease. What is the probability that a randomly selected individual with a positive test result has the decease? (10 p)

Problem 3

Let $X_1, \dots, X_n$ be independent copies of $X \sim f(x; \theta)$. For the MLE $\widehat{\theta}_n$ based on $X_1, \dots, X_n$ the convergence $\sqrt{n}(\widehat{\theta}_n - \theta) \stackrel{d}{\to} N(\mu, \sigma^2)$ as $n \to \infty$ holds (under mild regularity conditions).

(a) What is μ? what is σ^2? (5 p)

(b) From the convergence follows expressions for asymptotic/approximate mean and variance of the MLE $\widehat{\theta}_n$. Explain how the asymptotic/approximate variance relates to the Cramér-Rao lower bound. (5 p)

Problem 4

Consider data $x_1, \dots, x_n$ and a likelihood corresponding to the x_k, $k = 1, \dots, n$, being realizations of independent $\mathrm{Bin}(m, \theta)$-distributed random variables with probability mass function

$$x \mapsto \binom{m}{x} \theta^x (1-\theta)^{m-x}, \quad \theta \in (0,1).$$

(a) Show that a conjugate prior for θ is given by the $\mathrm{Beta}(\alpha, \beta)$ distribution with density function

$$\theta \mapsto \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}, \quad \alpha, \beta > 0$$

with prior mean $\mathrm{E}[\theta] = \alpha/(\alpha + \beta)$. Determine the parameters of the posterior distribution and its mean when the conjugate prior is used. (8 p)

(b) Determine the Jeffreys' prior for θ and determine the corresponding posterior distribution and its mean. (7 p)

- - - - - - - - **PART 2** - - - - - - - -

Problem 5

Let $X_1, \ldots, X_n$ be independent copies of $X \sim f(x;\theta)$.

(a) Explain what is meant by the power of a test with null hypothesis $H_0 : \theta = \theta_0$, and alternative hypothesis $H_1 : \theta = \theta_1$, with $\theta_1 \neq \theta_0$. (10 p)

(b) Let $X \sim N(\mu, \sigma^2)$, where $\sigma^2 > 0$ is known. Derive the power function of the 2-sided z-test for the null hypothesis $H_0 : \mu = \mu_0$. (15 p)

Problem 6

Let $X_1, \ldots, X_n$ be independent copies of $X \sim f(x;\theta)$. Let $S(\theta; X_1, \ldots, X_n)$ denote the score function based on $X_1, \ldots, X_n$. Show the convergence in distribution

$$\frac{S(\theta; X_1, \ldots, X_n)}{\sqrt{n}} \xrightarrow{d} N(\mu, \sigma^2) \text{ as } n \to \infty,$$

which includes determining μ and σ^2, with σ^2 expressed in terms of either Fisher information, observed Fisher information or expected Fisher information. (25 p)

Problem 1

The likelihood function

$$L(\theta; x_1, \ldots, x_k) = \prod_{k=1}^{n} \theta^{-1} \mathbb{1}_{[0,\theta]}(x_k) = \theta^{-n} \mathbb{1}_{[0,\theta]}\Big(\max_k x_k\Big)$$

is maximized by $\widehat{\theta}_n = \max_k x_k$. With $t = \max_k x_k$, $g_1(t;\theta) = \theta^{-n}\mathbb{1}_{[0,\theta]}(t)$ and $g_2(x_1, \ldots, x_n) = 1$ we can write

$$f(x_1, \ldots, x_k; \theta) = L(\theta; x_1, \ldots, x_k) = g_1(t, \theta) g_2(x_1, \ldots, x_n).$$

By the factorization theorem we have shown that $t = \max_k x_k$ is sufficient for θ.

Problem 2

$$\begin{aligned}
\mathrm{P}(D+ \mid T+) &= \frac{\mathrm{P}(T+ \mid D+)\,\mathrm{P}(D+)}{\mathrm{P}(T+ \mid D+)\,\mathrm{P}(D+) + \mathrm{P}(T+ \mid D-)\,\mathrm{P}(D-)} \\
&= \frac{\mathrm{P}(T+ \mid D+)\,\mathrm{P}(D+)}{\mathrm{P}(T+ \mid D+)\,\mathrm{P}(D+) + (1 - \mathrm{P}(T- \mid D-))\,\mathrm{P}(D-)} \\
&= \frac{0.9 \cdot 0.01}{0.9 \cdot 0.01 + (1 - 0.9) \cdot 0.99} = \frac{9}{9 + 99} = \frac{1}{12}
\end{aligned}$$

Problem 3

$\mu = 0$, $\sigma^2 = J_X(\theta)^{-1} = \mathrm{E}[I(\theta; X)]^{-1}$. In particular, $\widehat{\theta}_n$ is approximately/asymptotically unbiased with approximate variance $(nJ_X(\theta))^{-1}$, which coincides with the Cramér-Rao lower bound.

Problem 4

$$\begin{aligned}
f(\theta \mid x_1, \ldots, x_n) &\propto f(x_1, \ldots, x_n \mid \theta) f(\theta) \\
&= \left(\prod_{k=1}^{n} \binom{m}{x_k}\right) \frac{1}{B(\alpha, \beta)} \theta^{\sum_k x_k + \alpha - 1} (1 - \theta)^{nm - \sum_k x_k + \beta - 1}
\end{aligned}$$

which (after normalization) gives a posterior distribution which is a Beta distribution with parameters

$$\alpha' = \sum_k x_k + \alpha, \quad \beta' = nm - \sum_k x_k + \beta$$

and mean $\alpha'/(\alpha' + \beta')$.

The Binomial likelihood and log-likelihood are $L(\theta; x) = \binom{m}{x}\theta^x(1-\theta)^{m-x}$ and $l(\theta; x) = \log\binom{m}{x} + x \log\theta + (m - x)\log(1 - \theta)$. The score function is $S(\theta; x) = x/\theta - (m - x)/(1 - \theta)$. The Fisher information is $I(\theta; x) = x/\theta^2 + (m - x)/(1 - \theta)^2$. Expected Fisher is

$$J_X(\theta) = \mathrm{E}[I(\theta; X)] = \frac{m\theta}{\theta^2} + \frac{m - m\theta}{(1 - \theta)^2} = \frac{m}{\theta} + \frac{m}{1 - \theta} = \frac{m}{\theta(1 - \theta)}.$$

Jeffrey's prior density is proportional to $\sqrt{J_X(\theta)}$, i.e. to $\theta^{1/2-1}(1-\theta)^{1/2-1}$. Hence, Jeffrey's prior corresponds to Beta$(1/2, 1/2)$. Hence, the posterior is a Beta distribution with parameters

$$\alpha' = \sum_k x_k + 1/2, \quad \beta' = nm - \sum_k x_k + 1/2.$$

Problem 5

(a) The power is the probability of rejecting H_0 when $\theta = \theta_1$.
(b) The power function is the probability of rejecting H_0 as a function of the parameter μ. The z-statistic is $Z_n := (\overline{X}_n - \mu_0)/(\sigma/\sqrt{n})$. Under H_0 the z-statistic is Z_n is standard normal. Therefore, the rejection region is $R_\alpha = \{z : z < \Phi^{-1}(\alpha/2)\} \cup \{z : z > \Phi^{-1}(1-\alpha/2)\}$ if the probability of rejecting a true null hypothesis is chosen to be α. The power function is, with $G \sim N(0,1)$,

$$\begin{aligned}
g(\mu) &= \mathrm{P}(Z_n \in R_\alpha) \\
&= \mathrm{P}\left(\frac{\overline{Z}_n - \mu}{\sigma/\sqrt{n}} + \frac{\mu - \mu_0}{\sigma/\sqrt{n}} < \Phi^{-1}(\alpha/2)\right) + \mathrm{P}\left(\frac{\overline{Z}_n - \mu}{\sigma/\sqrt{n}} + \frac{\mu - \mu_0}{\sigma/\sqrt{n}} > \Phi^{-1}(1-\alpha/2)\right) \\
&= \mathrm{P}\left(G < \Phi^{-1}(\alpha/2) - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right) + \mathrm{P}\left(G > \Phi^{-1}(1-\alpha/2) - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right) \\
&= \Phi\left(\Phi^{-1}(\alpha/2) - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right) + 1 - \Phi\left(\Phi^{-1}(1-\alpha/2) - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right)
\end{aligned}$$

Problem 6

See e.g. the course book for details.