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Theory of Statistical Inference

Exam, 2022/10/24

The only allowed aid is a pocket calculator provided by the department. The answers to the tasks should be clearly formulated and structured. All non-trivial steps need to be commented. The solutions should be given in English or Swedish.

The written exam is divided into two parts. The first part considers the most central of the course concepts and it is related to standard problems. The second part consists of problems that requires a higher level of understanding, the ability to generalize and to combine methods. Each part consists of three problems and will worth a maximum of 50 points. In order to receive grades A-E, a minimum of 35 points is required in the first part. The second part is only graded for students passing the first part. Given a minimum of 35 points in the first part, the final grade is determined by the sum of regular points in both parts of the exam and bonus points according to the following table:

Grade	A	B	C	D	E	F
Points	$\geq$90	(90-80]	(79-70]	(69-60]	< 60 and ≥ 35 in Part I	< 35 in Part I

Up to 10 bonus points (i.e., in addition to the ordinary 100 points) are given for the active participation in the problem sessions. A half of the bonus points will be used for the first part of the exam, while the second half of the bonus points will be used in the second part of the exam.

Part I:

Problem 1 [23P]

Suppose that we have an iid sample $X_{1:n} = (X_1, X_2, ..., X_n)$ from a Pareto distribution with density of X_i given by

$$f_{X_i}(x; \beta) = \beta x^{-(\beta+1)} \qquad \text{for} \qquad x > 1$$

where $\beta > 0$ is an unknown parameter.

(a) Derive the maximum likelihood estimate $\hat{\beta}_{ML}$ for β. **[4P]**

(b) Derive the ordinary Fisher information $I_{1:n}(\beta)$, the observed Fisher information $I_{1:n}(\hat{\beta}_{ML})$, and the expected Fisher information $J_{1:n}(\beta)$. **[3P]**

(c) Find a minimum sufficient statistic for β and explain your answer.**[3P]**

(d) Construct a 95% two-sided score confidence interval for β. Simplified the expression of the confidence interval as much as possible. **[3P]**

(e) Construct a 95% two-sided Wald confidence interval for β. Simplified the expression of the confidence interval as much as possible. **[3P]**

(f) Construct a 95% two-sided likelihood ratio confidence interval for β. Simplified the expression of the confidence interval as much as possible. **[3P]**

(g) Does the value of the parameter $\beta_0 = 3$ lie in each of the three constructed confidence intervals, when $n = 36$ and $\sum_{i=1}^{n} \log(x_i) = 16.23$? **[4P]**

Hint: *Important quantiles of the standard normal distribution are:*

$z_{0.9}$	$z_{0.95}$	$z_{0.975}$	$z_{0.95}$
1.28	1.64	1.96	2.33

See Problem 4 for the important quantiles of the χ^2-distribution at various degrees of freedom.

Problem 2 [17P]

Let $X_{1:n} = (X_1, ..., X_n)$ denote a random sample from a geometric distribution with probability mass function of X_i given by

$$\mathbb{P}(X_i = x|\pi) = (1-\pi)^{x-1}\pi \qquad \text{for} \qquad x = 1, 2, ... \quad \text{and} \quad \pi \in [0, 1].$$

It also holds that $\mathbb{E}(X_i|\pi) = \frac{1}{\pi}$ and $\mathbb{V}ar(X_i|\pi) = \frac{1-\pi}{\pi^2}$.

(a) Show that the conjugate prior for π is given by the beta distribution with parameters $\alpha > 0$ and $\beta > 0$, that is

$$f(\pi) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \pi^{\alpha-1}(1-\pi)^{\beta-1} \quad \text{for} \quad \pi \in [0, 1] \quad \text{and} \quad \alpha, \beta > 0$$

with prior mean $\mathbb{E}(\pi) = \frac{\alpha}{\alpha+\beta}$. Determine the parameters in the corresponding posterior distribution. **[5P]**

(b) Compute the posterior mean of π when the conjugate prior is used. **[2P]**

(c) Find the expression of the Jeffreys prior for π and compute the corresponding posterior distribution. **[6P]**

(d) Calculate the posterior mean when the Jeffreys prior is used and compare it to the expression of the MLE estimator for π obtained in the frequentist statistics. **[4P]**

Problem 3 [10P]

Provide the definition of the expected Fisher information in the case of a scalar parameter. Derive the expectation and the variance of the score function in the case of a scalar parameter.

Part II:

Problem 4 [17P]

Let $X_{1:n_1} = (X_1, ..., X_{n_1})$ denote a random sample from a gamma distribution with density of X_i, $i = 1, ..., n_1$, given by

$$f_{X_i}(x;\theta_1) = \frac{\theta_1^k}{\Gamma(k)} x^{k-1} \exp(-\theta_1 x) \quad \text{for} \quad x > 0 \quad \text{and} \quad k, \theta_1 > 0$$

and let $X_{n_1+1:n_1+n_2} = (X_{n_1+1}, ..., X_{n_1+n_2})$ denote a random sample from a gamma distribution with density of X_j, $j = n_1 + 1, ..., n_1 + n_2$, given by

$$f_{X_j}(x;\theta_2) = \frac{\theta_2^k}{\Gamma(k)} x^{k-1} \exp(-\theta_2 x) \quad \text{for} \quad x > 0 \quad \text{and} \quad k, \theta_2 > 0$$

Assume that $X_{1:n_1}$ and $X_{n_1+1:n_1+n_2}$ are independent and let k be known.

The aim is to test the null hypothesis:

$$H_0: \ \theta_1 = \theta_2. \tag{1}$$

(a) Derive the generalized likelihood ratio statistic for testing H_0 in (1). Simplified the expression of the test statistic as much as possible. **[12P]**

(b) Determine the distribution of the test statistics derived in part (a). **[1P]**

(c) Perform the generalized likelihood ratio test at significance level of 5% when $k = 5$, $n_1 = 20$, $n_2 = 30$, $\sum_{i=1}^{n_1} x_i = 18.11$ and $\sum_{j=n_1+1}^{n_1+n_2} x_j = 36.24$. **[4P]**

Hint: *Important quantiles of the χ^2-distribution at various degrees of freedom are:*

d	1	2	3	4	5
$\chi^2_{0.9}(\mathrm{df} = d)$	2.71	4.61	6.25	7.78	9.24
$\chi^2_{0.95}(\mathrm{df} = d)$	3.84	5.99	7.81	9.49	11.07
$\chi^2_{0.975}(\mathrm{df} = d)$	5.02	7.38	9.35	11.14	12.83

Problem 5 [18P]

Let $\mathbf{X} = (X_1, X_2, X_3)^\top$ be independent random variable which is negative multinomially distributed with probability mass function expressed as

$$\mathbb{P}(\mathbf{X} = \mathbf{x}; \pi_1, \pi_2, \pi_3) = \frac{(n + x_1 + x_2 + x_3 - 1)!}{(n-1)!x_1!x_2!x_3!}(1 - \pi_1 - \pi_2 - \pi_3)^n \pi_1^{x_1}\pi_1^{x_2}\pi_3^{x_3}$$

for $\mathbf{x} = (x_1, x_2, x_3)^T$, $x_1, x_2, x_3 = 0, 1, 2, ...$, where $\mathbf{X}$ can be interpreted as the numbers of successes for three classes until n failures (none of the three classes is chosen) are observed, where k is known. The success probabilities of the three classes are denoted by $\pi_1 \in (0, 1)$, $\pi_2 \in (0, 1)$, and $\pi_3 \in (0, 1)$, respectively. Consequently, $\pi_0 = 1 - \pi_1 - \pi_2 - \pi_3$ is the probability of failure. Furthermore, it holds that $\mathbb{E}(X_i) = \frac{n}{\pi_0}\pi_i$ for $i = 1, 2, 3$.

(a) Derive the maximum likelihood estimator $\hat{\boldsymbol{\pi}}_{ML}$ for $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)^\top$. **[4P]**

(b) Compute the expected Fisher information $\mathbf{J}(\boldsymbol{\pi})$. **[4P]**

(c) Construct a 90% Wald confidence region for $\boldsymbol{\pi}$. Simplified the expression of the confidence region as much as possible.**[4P]**

(d) Construct a 90% likelihood ratio confidence region for $\boldsymbol{\pi}$. Simplified the expression of the confidence region as much as possible.**[2P]**

(e) Does the parameter vector $\boldsymbol{\pi}_0 = (0.15, 0.2, 0.35)^\top$ lie in each of the two constructed confidence regions, when $n = 27$, $x_1 = 6$, $x_2 = 12$ and $x_3 = 15$? **[4P]**

Hint: *Important quantiles of the χ^2-distribution at various degrees of freedom are:*

d	1	2	3	4	5
$\chi^2_{0.9}(\text{df} = d)$	2.71	4.61	6.25	7.78	9.24
$\chi^2_{0.95}(\text{df} = d)$	3.84	5.99	7.81	9.49	11.07
$\chi^2_{0.975}(\text{df} = d)$	5.02	7.38	9.35	11.14	12.83

Problem 6 [15P]

Prove that under Fisher regularity conditions the maximum likelihood estimator of a scalar parameter is asymptotically normally distributed.