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Theory of Statistical Inference

Exam, 2024/12/13

The only allowed aid is a pocket calculator provided by the department. The answers to the tasks should be clearly formulated and structured. The answers should be simplified as far as possible. All non-trivial steps need to be commented. The solutions should be given in English or Swedish.

The written exam is divided into two parts. The first part considers some central concepts of the course. The second part consists of problems that require a higher level of understanding, the ability to generalize and to combine methods. Each part will be worth a maximum of 50 points. In order to receive grades A-E, a minimum of 35 points is required in the first part. The second part is only graded for students passing the first part. Given a minimum of 35 points in the first part, the final grade is determined by the sum of points in both parts of the exam and bonus points according to the following table:

Grade	A	B	C	D	E	F
Points	$\geq$90	[80,90)	[70,80)	[60,70)	< 60 and ≥ 35 in Part I	< 35 in Part I

Up to 10 bonus points (i.e., in addition to the regular 100 points) are given for the active participation in the problem sessions. The bonus points will be used for the first part of the exam.

Please number all sheets of paper that you hand in, so that their order is easy to recover (just in case).

Lycka till!

Part I:

Problem 1 [5P]

Let X be uniformly distributed on $[0, \theta]$.

(a) Prove that $Z = X/\theta$ is a pivot for θ. Derive the distribution of Z. **[2P]**

(b) Construct a two-sided $(1-\alpha)$ confidence interval for the parameter θ. **[2P]**

(c) Let Y be uniformly distributed on $[\theta, 1]$. Find a pivot for parameter θ and derive its distribution. **[1P]**

Problem 2 [17P]

Let $X_{1:n} = (X_1, ..., X_n)$ be an iid sample from a normal distribution with density of X_i, $i = 1, ..., n$, given by

$$f_{X_i}(x; \mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad \text{for} \quad x \in \mathbb{R},\ \mu \in \mathbb{R},$$

where $\sigma^2 > 0$ is assumed to be known.

(a) Show that the conjugate prior for μ is given by the normal distribution with parameters $\mu_0 \in \mathbb{R}$ and $\sigma_0^2 > 0$ and specify the parameters of the corresponding posterior distribution. **[4P]**

(b) Compute two Bayesian estimators for μ when the conjugate prior is used. **[3P]**

(c) Derive the expression of a 95% credible interval for μ when the conjugate prior is used. **[2P]**

(d) Find the expression of the Jeffreys prior for μ and compute the corresponding posterior distribution. **[4P]**

(e) Compute two Bayesian estimators and a 95% credible interval for μ when the Jeffreys prior is used. **[4P]**

Problem 3 [22P]

Let $X_{1:n} = (X_1, X_2, ..., X_n)$ be an iid sample from a Borel distribution with density of X_i given by

$$\mathbb{P}(X_i = x; \theta) = \frac{1}{x!}(\theta x)^{x-1} \exp(-\theta x), \qquad x = 1, 2, \ldots$$

where $\theta \in (0, 1)$ is an unknown parameter. It also holds that

$$\mathbb{E}[X_i] = (1-\theta)^{-1} \quad \text{and} \quad \text{Var}(X_i) = \theta(1-\theta)^{-3}.$$

(a) Derive the maximum likelihood estimator $\hat{\theta}_{ML}$ for θ. **[4P]**

(b) Derive the (ordinary) Fisher information $I_{1:n}(\theta)$, the observed Fisher information $I_{1:n}(\hat{\theta}_{ML})$, and the expected Fisher information $J_{1:n}(\theta)$. **[4P]**

(c) Find a minimal sufficient statistic for θ and explain your answer. **[3P]**

(d) Determine the asymptotic distribution of the properly normalized $\hat{\theta}_{ML}$ as $n \to \infty$ and provide the analytical expressions of its parameters as functions of θ. **[3P]**

(e) Construct a 95% two-sided Wald confidence interval for θ. **[3P]**

(f) Based on the observed data

$$x_{1:12} = (2, 7, 5, 3, 2, 9, 2, 3, 5, 6, 4, 4),$$

perform the Wald test of the hypothesis

$$H_0 : \theta = 0.5 \quad \text{versus} \quad H_1 : \theta \neq 0.5$$

at significance level of 5%. **[4P]**

Hint: *Important quantiles of the standard normal distribution are:*

$z_{0.9}$	$z_{0.95}$	$z_{0.975}$	$z_{0.995}$
1.28	1.64	1.96	2.33

Problem 4 [6P]

Provide the expression of the likelihood ratio (LR) statistic in the case of a scalar parameter. What is the asymptotic distribution of the LR statistic? Describe how a likelihood ratio confidence interval is constructed.

Part II:

Problem 5 [16P]

Let $X_{1:n_1} = (X_1, ..., X_{n_1})$ be an iid sample from a normal distribution with density of X_i, $i = 1, ..., n_1$, given by

$$f_{X_i}(x; \mu_1, \sigma_1^2) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) \quad \text{for} \quad x \in \mathbb{R}, \mu_1 \in \mathbb{R}, \sigma_1^2 > 0$$

and let $X_{n_1+1:n_1+n_2} = (X_{n_1+1}, ..., X_{n_1+n_2})$ be an iid sample from a normal distribution with density of X_j, $j = n_1 + 1, ..., n_1 + n_2$, given by

$$f_{X_j}(x; \mu_2, \sigma_2^2) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right) \quad \text{for} \quad x \in \mathbb{R}, \mu_2 \in \mathbb{R}, \sigma_2^2 > 0$$

Assume that $X_{1:n_1}$ and $X_{n_1+1:n_1+n_2}$ are independent.

The aim is to test the null hypothesis:

$$H_0 : \ \sigma_1^2 = \sigma_2^2. \tag{1}$$

(a) Derive the generalized likelihood ratio statistic for testing H_0 in (1). Simplify the expression of the test statistic as much as possible. **[11P]**

(b) Determine the distribution of the test statistic derived in part (a). **[1P]**

(c) Perform the generalized likelihood ratio test at significance level of 5% when $n_1 = 15$, $n_2 = 25$, $\sum_{i=1}^{n_1} x_i = 12$, $\sum_{j=n_1+1}^{n_1+n_2} x_j = 33$, $\sum_{i=1}^{n_1}(x_i - \bar{x}_1)^2 = 14$ and $\sum_{j=n_1+1}^{n_1+n_2}(x_j - \bar{x}_2)^2 = 51$ where $\bar{x}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i$ and $\bar{x}_2 = \frac{1}{n_2}\sum_{j=n_1+1}^{n_1+n_2} x_j$. **[4P]**

Hint: *Important quantiles of the χ^2-distribution at various degrees of freedom are:*

d	1	2	3	4	5
$\chi^2_{0.9}(\mathrm{df}=d)$	2.71	4.61	6.25	7.78	9.24
$\chi^2_{0.95}(\mathrm{df}=d)$	3.84	5.99	7.81	9.49	11.07
$\chi^2_{0.975}(\mathrm{df}=d)$	5.02	7.38	9.35	11.14	12.83

Problem 6 [18P]

Let $X_{1:n} = (X_1, ..., X_n)$ be an iid sample from a Lomax distribution with density of X_i given by

$$f_{X_i}(x;\beta) = \beta(1+x)^{-(\beta+1)} \qquad \text{for} \qquad x \geq 0\,,$$

where $\beta > 0$ is an unknown parameter.

(a) Derive the maximum likelihood estimator $\hat{\beta}_{ML}$ for β. **[3P]**

(b) Derive the ordinary Fisher information $I_{1:n}(\beta)$ and the expected Fisher information $J_{1:n}(\beta)$. **[3P]**

(c) Determine the asymptotic distribution of $\hat{\beta}_{ML}$. Simplify the expression of the asymptotic distribution as much as possible.**[2P]**

(d) Specify the variance stabilising transformation $\phi = h(\beta)$. **[3P]**

(e) Derive the maximum likelihood estimator for ϕ and determine its asymptotic distribution. Simplify the expression of the asymptotic distribution as much as possible. **[3P]**

(f) Using the result of part (e), find a 95% confidence interval for β.**[4P]**

Problem 7 [16P]

Let $X_{1:n} = (X_1, ..., X_n)$ be an independent sample from a normal distribution with $X_i \sim N(\mu, \sigma^2 u_i^2)$, i.e., the density of X_i, $i = 1, ..., n$ is given by

$$f_{X_i}(x_i;\mu) = \frac{1}{\sqrt{2\pi\sigma^2 u_i^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2 u_i^2}\right) \quad \text{for} \quad x_i \in \mathbb{R},\ \mu \in \mathbb{R}, \sigma > 0,$$

where μ and σ^2 are unknown model parameters and $\{u_i^2\}$ is a sequence of known positive quantities.

(a) Derive the Jeffreys prior for μ and σ^2.**[6P]**

(b) Determine the posterior for μ and σ^2 when the Jeffreys prior is used.**[3P]**

(c) Derive a Bayesian estimator for μ and σ^2 when the Jeffreys prior is used.**[4P]**

(d) Derive the marginal posterior distribution of σ^2 when the Jeffreys prior is used. **[3P]**