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Theory of Statistical Inference

Exam, 2024/10/28

The only allowed aid is a pocket calculator provided by the department. The answers to the tasks should be clearly formulated and structured. The answers should be simplified as far as possible. All non-trivial steps need to be commented. The solutions should be given in English or Swedish.

The written exam is divided into two parts. The first part considers some central concepts of the course. The second part consists of problems that require a higher level of understanding, the ability to generalize and to combine methods. Each part will be worth a maximum of 50 (regular) points. In addition there is a bonus question at the end of the second part with which you could earn up to 5 extra points. In order to receive grades A-E, a minimum of 35 points is required in the first part. The second part is only graded for students passing the first part. Given a minimum of 35 points in the first part, the final grade is determined by the sum of regular points in both parts of the exam and bonus points according to the following table:

Grade	A	B	C	D	E	F
Points	$\geq$90	(90-80]	(79-70]	(69-60]	< 60 and ≥ 35 in Part I	< 35 in Part I

Up to 10 bonus points (i.e., in addition to the regular 100 points) are given for the active participation in the problem sessions. The bonus points will be used for the first part of the exam.

Please number all sheets of paper that you hand in, so that their order is easy to recover (just in case). On the first page please state how many bonus points you have obtained. If you don't know, simply put the best possible lower bound followed by +, e.g., 8+.

Lycka till!

Part I:

Problem 1 [9P]

(a) Provide the definitions of a sufficient statistic and of a minimal sufficient statistic. Formulate the statement of the factorization theorem. **[5P]**

(b) Consider and iid sample $X_{1:n} = (X_1, \dots, X_n)$ from a Bernoulli distribution with probability mass function given by

$$P(X_i = 1) = p \quad \text{and} \quad P(X_i = 0) = 1 - p \quad \text{for } p \in (0, 1).$$

Find a minimal sufficient statistic for p. **[2P]**

(c) Consider and iid sample $X_{1:n} = (X_1, \dots, X_n)$ from a beta distribution with density given by

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad \alpha, \beta > 0.$$

Find a minimal sufficient statistic for $(\alpha, \beta)^\top$. **[2P]**

Problem 2 [9P]

Let $X_{1:n} = (X_1, \dots, X_n)$ be an iid sample from a log-normal distribution with density given by

$$f(x; \boldsymbol{\theta}) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma^2}\right) \quad \text{for} \quad x > 0 \text{ and } \boldsymbol{\theta} = (\mu, \sigma^2)^\top \text{ with } \mu \in \mathbb{R}, \sigma > 0.$$

(a) Derive the expression of the log-likelihood function $\ell(\boldsymbol{\theta})$. **[2P]**

(b) Calculate the score vector $S(\boldsymbol{\theta})$. **[3P]**

(c) Using the properties of the score vector, find $E[(\log X_1)^2]$. **[4P]**

Problem 3 [13P]

Let $X_{1:n} = (X_1, ..., X_n)$ denote a random sample from a Maxwell–Boltzmann distribution with density of X_i given by

$$f_{X_i}(x; \theta) = \sqrt{\frac{2}{\pi}} \theta^{3/2} x^2 \exp\left(-\theta \frac{x^2}{2}\right) \qquad \text{for} \qquad x > 0 \quad \text{and} \quad \theta > 0.$$

(a) Show that the conjugate prior for θ is given by the gamma distribution with hyperparameters $\alpha > 0$ and $\beta > 0$, that is

$$f(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} \exp(-\beta\theta) \quad \text{for} \quad \theta > 0 \quad \text{and} \quad \alpha, \beta > 0$$

with prior mean $\mathbb{E}(\theta) = \frac{\alpha}{\beta}$. Determine the parameters in the corresponding posterior distribution. **[5P]**

(b) Compute two Bayesian point estimators of θ when the conjugate prior is used. **[4P]**

(c) Find the expression of the Jeffreys prior for θ and compute the corresponding posterior distribution. **[4P]**

Problem 4 [19P]

Let $X_{1:n} = (X_1, X_2, ..., X_n)$ be an iid sample from a gamma distribution with density of X_i given by

$$f_{X_i}(x;\theta) = \frac{\theta^\beta}{\Gamma(\beta)} x^{\beta-1} \exp(-\theta x) \qquad \text{for} \qquad x > 0, \;\; \theta > 0,$$

where $\beta > 0$ is assumed to be known and $\Gamma(\cdot)$ denotes the gamma function.

(a) Derive the maximum likelihood estimator $\hat{\theta}_{ML}$ for θ. **[5P]**

(b) Derive the (ordinary) Fisher information $I_{1:n}(\theta)$, the observed Fisher information $I_{1:n}(\hat{\theta}_{ML})$, and the expected Fisher information $J_{1:n}(\theta)$. **[4P]**

(c) Derive the test statistic of the score test for the null hypothesis $H_0 : \theta = 1$. Simplify the expression of the test statistic as much as possible. **[4P]**

(d) Derive the test statistic of the Wald test for the null hypothesis $H_0 : \theta = 1$. Simplify the expression of the test statistic as much as possible. **[4P]**

(e) Perform the score test and the Wald test at the significance level of 5%, when $\beta = 4$, $n = 25$ and $\sum_{i=1}^n x_i = 81.7$? **[2P]**

Hint: *Important quantiles of the standard normal distribution are:*

$z_{0.9}$	$z_{0.95}$	$z_{0.975}$	$z_{0.995}$
1.28	1.64	1.96	2.33

Part II:

Problem 5 [18P]

Let $X_{1:n} = (X_1, \dots, X_n)$ be an iid sample from a normal distribution with density of X_i, $i = 1, ..., n$, given by

$$f_{X_i}(x;\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad \text{for} \quad x \in \mathbb{R}, \; \mu \in \mathbb{R}, \sigma > 0$$

(a) Derive the likelihood ratio statistic for the test on $H_0 : \mu = \mu_0$ when σ is known. What is the asymptotic distribution of the derived test statistic under the null hypothesis? **[7P]**

(b) Perform the test from part (a) at significance level of 10%, when $n = 50$, $\sigma^2 = 1$, $\mu_0 = 0$, and $\sum_{i=1}^n x_i = 10$. **[2P]**

(c) Derive the likelihood ratio statistic for the test on $H_0 : \sigma^2 = \sigma_0^2$ when μ is known. What is the asymptotic distribution of the derived test statistic under the null hypothesis? **[7P]**

(d) Perform the test from part (c) at significance level of 10%, when $n = 50$, $\sigma_0^2 = 1$, $\mu = 0$, and $\sum_{i=1}^n x_i^2 = 80$. **[2P]**

Hint: *Important quantiles of the χ^2-distribution at various degrees of freedom are:*

d	1	2	3	4	5
$\chi^2_{0.9}(\text{df} = d)$	2.71	4.61	6.25	7.78	9.24
$\chi^2_{0.95}(\text{df} = d)$	3.84	5.99	7.81	9.49	11.07
$\chi^2_{0.975}(\text{df} = d)$	5.02	7.38	9.35	11.14	12.83

Problem 6 [17P]

Let $X_{1:n} = (X_1, ..., X_n)$ be an iid sample from a Poisson distribution with probability mass function of X_i, $i = 1, ..., n$, given by

$$\mathbb{P}(X_i = x; \lambda) = \frac{\lambda^x}{x!} \exp(-\lambda) \quad \text{for} \quad \lambda > 0 \quad \text{and} \quad x = 0, 1, 2, ...$$

Furthermore, it holds that $\mathbb{E}(X_i) = \lambda$.

(a) Derive the maximum likelihood estimator $\hat{\lambda}_{ML}$ for λ. **[3P]**

(b) Derive the (ordinary) Fisher information $I_{1:n}(\lambda)$ and the expected Fisher information $J_{1:n}(\lambda)$. **[3P]**

(c) Determine the asymptotic distribution of $\hat{\lambda}_{ML}$. Simplify the expression of the asymptotic distribution as much as possible. **[2P]**

(d) Specify the variance stabilising transformation $\phi = h(\lambda)$. **[3P]**

(e) Derive the maximum likelihood estimator for ϕ and determine its asymptotic distribution. Simplify the expression of the asymptotic distribution as much as possible. **[3P]**

(f) Using the result of part (e), find a 95% confidence interval for λ. **[3P]**

Problem 7 [15P]

Formulate the statement of the Cramér-Rao lower bound and prove it.

Bonus question [5P]

In Problem 1 you formulated the statement of the factorization theorem. Prove the statement of the factorization theorem.