MT7047 – Probability theory III – exam

Date Thursday November 30, 2023
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Tools None.
Grading criteria The exam consists of two parts, which consist of 20 and 40 points respectively. To pass the exam a score of 14 or higher is required on Part I. If attained, then also Part II is graded, and the score on this part determines the grade. Grades are determined according to the following table:

	A	B	C	D	E
Part I	14	14	14	14	14
Part II	32	24	16	8	0

Problems of Part I may give up to five points each, and problems of Part II may give up to ten points each. Complete and well motivated solutions are required for full score. Partial solution may be rewarded with a partial score.

Part I

Problem 1. Let X and Y be random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that $Z = \max\{X, Y\}$ is also a random variable.

Problem 2. Toss a fair coin five times.

(a) Construct a probability space corresponding to the above experiment.

(b) Let X denote the number of tosses that turn out 'heads' and compute $\mathbb{E}[X]$ using the definition of expectation.

Problem 3. Let $X_1, X_2, \ldots$ be independent random variables taking values ± 1 with equal probability, and let $g : \mathbb{R} \to [0, 1]$ be some function. Set $Y_0 = 0$ and for $n \geq 0$ let $Y_{n+1} - Y_n = g(Y_n)X_{n+1}$. Show that $(Y_n)_{n\geq 0}$ defines a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$, where $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

Problem 4. Let $X_1, X_2, \ldots$ be random variabels on some probability space such that $|X_n| \leq K$ for some $K < \infty$ and all $n \geq 1$. Suppose that X is another random variable and that $X_n \xrightarrow{p} X$ as $n \to \infty$. Show that $\mathbb{P}(|X| \leq K) = 1$.

Part II

Problem 5. Let $(X_n)_{n\geq 1}$ be a square-integrable martingale with respect to some filtration $(\mathcal{F}_n)_{n\geq 1}$.

(a) Show that $\mathbb{E}[X_n] = \mathbb{E}[X_1]$ for all $n \geq 1$.

(b) Show that $\mathbb{E}[X_n^2]$ is nondecreasing in n.

Problem 6. Let $X_1, X_2, \ldots$ be independent Bernoulli-distributed random variables with parameter $p = 1/2$. Let $S_n = X_1 + \ldots + X_n$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and set $S_0 = 0$.

(a) Show that $(S_n - n/2)_{n \geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n \geq 0}$.

(b) Let $T = \min\{n \geq 1 : S_n \geq 10\}$ and determine $\mathbb{E}[T]$.

Problem 7. Consider the probability space $([0,1], \mathcal{B}[0,1], \mathbb{P})$, where $\mathcal{B}[0,1]$ denotes the Borel σ-algebra on $[0,1]$ and $\mathbb{P}$ denotes Lebesgue measure. Let

$$X(\omega) = \begin{cases} 2\omega & \text{for } \omega \in [0, 1/2), \\ 1 & \text{for } \omega \in [1/2, 1], \end{cases}$$

and

$$Y(\omega) = \begin{cases} 0 & \text{for } \omega \in [0, 1/2), \\ 1 & \text{for } \omega \in [1/2, 1]. \end{cases}$$

(a) Determine the distribution function $F_X(x) = \mathbb{P}(X \leq x)$ for $x \in [0,1]$.

(b) Determine the conditional probability $\mathbb{P}(X \leq x \,|\, Y)$ for $x \in [0,1]$.

Problem 8. Consider an urn which initially contains one red and one blue ball. In round $k \geq 1$ a ball is drawn uniformly at random, and replaced together with 2^k balls of the same colour.

(a) Let N denote the number of times the initial red ball is drawn. Show that $\mathbb{E}[N] = 1$ and that $\mathbb{P}(N = 0) > 0$.

(b) Let Y_n denote the proportion of red balls in the urn after n rounds. Show that the limit $Y = \lim_{n \to \infty} Y_n$ exists, and that $\mathbb{P}(Y = 0) > 0$.