Solutions exam Probability theory III 2023-11-30

$\boxed{1}$ A function $X : \Omega \to \mathbb{R}$ is a random variable if for every Borelset $B \in \mathcal{B}(\mathbb{R})$ we have

$$\{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$ 2p

By proposition it suffices to consider sets of the form $(-\infty, x]$.

We have for every $x \in \mathbb{R}$ that

$$\{\omega \in \Omega : \max\{X(\omega), Y(\omega)\} \leq x\}$$
$$= \{\omega \in \Omega : X(\omega) \leq x\} \cap \{\omega \in \Omega : Y(\omega) \leq x\}$$

The latter two are in $\mathcal{F}$, since X and Y rvs, and hence their intersection. So, $\max\{X, Y\}$ is a rv. 3p

$\boxed{2}$ (a) The following is suitable for a fair coin:

$\Omega = \{0,1\}^5$

$\mathcal{F} =$ power set of Ω

$\mathbb{P}(\omega) = \frac{1}{2^5}$ for every $\omega \in \Omega$. 2p

(b) Let $X(\omega) = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5$.
X takes values $0, 1, 2, 3, 4, 5$ and is therefore a simple random variable. By definition

$$\mathbb{E}[X] = \sum_{k=0}^{5} k \cdot \mathbb{P}(X = k).$$

Note that for any $A \subseteq \Omega$ we have

$$\mathbb{P}(A) = \frac{|A|}{2^5}$$

Since $|\{X = k\}| = \binom{5}{k}$ we obtain 3p

$$\mathbb{E}[X] = \sum_{k=1}^{5} k \binom{5}{k} \frac{1}{2^5} = \frac{1}{2^5}\left[1 \cdot 5 + 2 \cdot 10 + 3 \cdot 10 + 4 \cdot 5 + 5 \cdot 1\right]$$
$$= \frac{80}{32} = \frac{5}{2}.$$

$\boxed{3}$ First note that

$$|Y_{n+1}| \le |Y_n| + g(Y_n) \le |Y_n| + 1 \le \dots \le n+1. \qquad 1p$$

Moreover, Y_{n+1} is determined by Y_n and X_{n+1}. Hence, if $Y_n \in \mathcal{F}_n$, then $Y_{n+1} \in \mathcal{F}_{n+1}$. Since $Y_1 = g(0) \cdot X_1 \in \mathcal{F}_1$, it follows by induction that $Y_n \in \mathcal{F}_n$ for all n. 2p

Finally,

$$\mathbb{E}[Y_{n+1} - Y_n \mid \mathcal{F}_n] = g(Y_n)\, \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = 0. \qquad 2p$$

Hence the martingale property also holds.

$\boxed{4}$ Suppose $|X_n| \le K$ for all n and that $X_n \xrightarrow{P} X$. We have

$$\mathbb{P}(|X| > K + \varepsilon) \le \underbrace{\mathbb{P}(|X - X_n| > \varepsilon)}_{\to 0 \text{ as } n \to \infty} + \underbrace{\mathbb{P}(|X_n| > K)}_{=0} \qquad 2p$$

Since LHS does not depend on n, we have

$$\mathbb{P}(|X| > K + \varepsilon) = 0 \qquad \text{for all } \varepsilon > 0.$$

Finally, by continuity of measure,

$$\mathbb{P}(|X| > K) = \mathbb{P}\Big(\bigcup_{n \ge 1} \underbrace{\{|X| > K + \tfrac{1}{n}\}}_{\text{increasing in } n}\Big) = \lim \mathbb{P}(|X| > K + \tfrac{1}{n}) = 0. \qquad 3p$$

$\boxed{5}$ (a) Since $(X_n)_{n \ge 1}$ is a martingale, we have

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n \qquad \text{for all } n \ge 1. \qquad 2p$$

Taking expectations gives

$$\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n] \qquad \text{for all } n \ge 1.$$

Iteration gives $\mathbb{E}[X_n] = \mathbb{E}[X_{n-1}] = \dots = \mathbb{E}[X_1]$. 2p

(b) We first have

$$\mathbb{E}[X_{n+1}^2] = \mathbb{E}[(X_{n+1} - X_n + X_n)^2]$$

$$= \underbrace{\mathbb{E}[(X_{n+1} - X_n)^2]}_{\ge 0} + 2\,\mathbb{E}[(X_{n+1} - X_n) X_n] + \mathbb{E}[X_n^2]. \qquad 3p$$

Using the martingale property, and properties of cond. exp.

$$\mathbb{E}[(X_{n+1}-X_n)X_n] = \mathbb{E}[X_n \mathbb{E}[X_{n+1}-X_n \mid \mathcal{F}_n]] = 0.$$

This gives $\mathbb{E}[X_{n+1}^2] \geq \mathbb{E}[X_n^2]$ for all $n \geq 1$. 3p

$\boxed{6}$ (a) First S_n is a function of $X_1, \dots, X_n$, so clearly $\mathcal{F}_n$-measurable. Also $|X_k| \leq 1$, so $|S_n| \leq n + \frac{n}{2} < \infty$. Finally,

$$\mathbb{E}\left[S_{n+1} - \frac{n+1}{2} \mid \mathcal{F}_n\right] = S_n - \frac{n+1}{2} + \underbrace{\mathbb{E}[X_{n+1}]}_{=1/2} = S_n - \frac{n}{2},$$

so $(S_n - \frac{n}{2})_{n \geq 0}$ is a martingale. 3p

(b) We first note that

$$\{T=n\} = \{S_1 < 10\} \cap \dots \cap \{S_{n-1} < 10\} \cap \{S_n \geq 10\}$$

which is in $\mathcal{F}_n$, so T is a stopping time wrt $(\mathcal{F}_n)_{n \geq 1}$. 2p

We note that $S_{n+1} - S_n \in \{0, 1\}$, so in each step we increase by one or stay. It follows that $S_T = 10$. We want to use optional stopping, which would give

$$\mathbb{E}\left[S_T - \frac{T}{2}\right] = \mathbb{E}[S_0] = 0,$$

and hence $\mathbb{E}[T] = 2\mathbb{E}[S_T] = 20$. 2p

We verify the conditions of Optional stopping III:

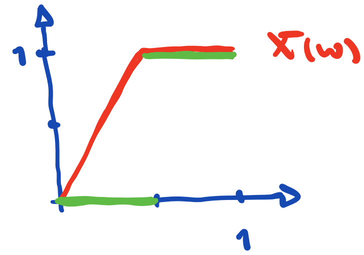
$$\mathbb{E}\left[\left| S_{n+1} - S_n + \tfrac{1}{2} \right| \mid \mathcal{F}_n\right] = \frac{1}{2}$$

and T denotes the number of "trials" before seeing a variable X_n taking value 1 for the 10th time. Hence T has a negative binomial distribution, whose expectation is finite. 3p

Alt. Let $T_k = \min\{n \geq 0 : S_n = k\}$. Then, $T_{k+1} - T_k$ has a geometric/first success-distribution with mean 2. Hence

$$\mathbb{E}[T] = \mathbb{E}[T_{10}] = \sum_{k=0}^{9} \mathbb{E}[T_{k+1} - T_k] = 2 \cdot 10 = 20.$$

7 Let's first make a figure:



(a) We have

$$F_X(x) = \mathbb{P}(X \le x)$$

$$= \begin{cases} \mathbb{P}(X \le 1) = \mathbb{P}([0,1]) = 1 & \text{if } x = 1 \\ \mathbb{P}(2\omega \le x) = \mathbb{P}([0, \frac{x}{2}]) = \frac{x}{2} & \text{if } x \in [0,1). \end{cases}$$

4p

(b) We note that Y takes the values 0 and 1 with equal probability. Hence Y is a simple random variable. By def, the conditional probability $\mathbb{P}(X \le x \mid Y)$ is thus the random variable such that

$$\mathbb{P}(X \le x \mid Y) = \begin{cases} \mathbb{P}(X \le x \mid Y = 1) & \text{on } \{Y = 1\} \\ \mathbb{P}(X \le x \mid Y = 0) & \text{on } \{Y = 0\} \end{cases}$$

3p

For $x = 1$ both expressions above equal 1. For $x \in [0,1)$

$$\mathbb{P}(X \le x \mid Y = 1) = \frac{\mathbb{P}(X \le x, Y = 1)}{\mathbb{P}(Y = 1)} = \frac{\mathbb{P}([0, \frac{x}{2}] \cap [\frac{1}{2}, 1])}{1/2} = 0.$$

3p

$$\mathbb{P}(X \le x \mid Y = 0) = \frac{\mathbb{P}(X \le x, Y = 0)}{\mathbb{P}(Y = 0)} = \frac{\mathbb{P}([0, \frac{x}{2}] \cap [0, \frac{1}{2}))}{1/2} = \frac{x/2}{1/2} = x.$$

8 (a) Initially there are two balls, and in round k we add 2^k balls. We claim that at the time of the n^{th} round, there are 2^n balls to choose from. Note that true for $n = 1$. Suppose true at round $n-1$. Then at round n there are

$$\underbrace{2^{n-1}}_{\text{present round } n-1} + \overbrace{2^{n-1}}^{\text{added round } n-1} = 2 \cdot 2^{n-1} = 2^n$$

2p

balls present. The claim follows from the induction principle.

Let $A_n = \{\text{initial red drawn in round } n\}$. Then $N = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}$ and

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} \overset{\text{geometric series}}{=} 1.$$

Next, note that

$$\mathbb{P}(N\geq 2) \geq \mathbb{P}(A_1\cap A_2) = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8} > 0.$$

Suppose $\mathbb{P}(N=0)=0$. Then

3p

$$\mathbb{E}[N] = \sum_{n\geq 0} n\cdot\mathbb{P}(N=n) \geq \mathbb{P}(N=1)+2\cdot\mathbb{P}(N\geq 2)$$

$$= \underbrace{\mathbb{P}(N\geq 1)}_{=1 \text{ if } \mathbb{P}(N=0)=0} + \mathbb{P}(N\geq 2) = 1 + \underbrace{\mathbb{P}(N\geq 2)}_{>0} > 1$$

This would contradict that $\mathbb{E}[N]=1$. Hence $\mathbb{P}(N=0)>0$.

(b) We want to show that $(Y_n)_{n\geq 0}$ is a martingale wrt itself. Then, since bounded, it follows from the martingale conv. thm. that

$$Y = \lim_{n\to\infty} Y_n$$

exists almost surely. We then also note that on the event that $N=0$, no red balls are ever added to the urn, in which case $Y=0$. It follows that

3p

$$\mathbb{P}(Y=0) \geq \mathbb{P}(N=0) > 0.$$

So, it remains to show that $(Y_n)_{n\geq 0}$ has the martingale prop. Let $X_n = \mathbb{1}\{\text{red in round } n\}$, and set $R_n = 1+\sum_{k=1}^{n} 2^k X_k$. Then

$$Y_n = \frac{R_n}{2^{n+1}}.$$

We have

2p

$$\mathbb{E}[Y_n - Y_{n-1} \mid \mathcal{F}_{n-1}] = \mathbb{E}\left[\frac{R_n - 2R_{n-1}}{2^{n+1}} \,\Big|\, \mathcal{F}_{n-1}\right]$$

$$= \mathbb{E}\left[\frac{2^n X_n - R_{n-1}}{2^{n+1}} \,\Big|\, \mathcal{F}_{n-1}\right]$$

$$= \frac{1}{2}\underbrace{\mathbb{E}[X_n \mid \mathcal{F}_{n-1}]}_{Y_{n-1}} - \frac{1}{2}Y_{n-1} = \frac{1}{2}Y_{n-1} - \frac{1}{2}Y_{n-1} = 0.$$