MT7047 – Probability theory III – exam

Date Friday November 1, 2024
Examiner Daniel Ahlberg
Tools None.
Grading criteria The exam consists of two parts, which consist of 20 and 40 points respectively. To pass the exam a score of 14 or higher is required on Part I. If attained, then also Part II is graded, and the score on this part determines the grade. Grades are determined according to the following table:

	A	B	C	D	E
Part I	14	14	14	14	14
Part II	32	24	16	8	0

Problems of Part I may give up to five points each, and problems of Part II may give up to ten points each. Complete and well motivated solutions are required for full score. Partial solution may be rewarded with a partial score.

Part I

Problem 1. A (European) roulette wheel consists of 37 equally spaced pockets, numbered from 0 to 36, of which 18 are red, 18 black and one green. Spinn the wheel four times and register the outcome.

(a) Construct a probability space corresponding to the above experiment.

(b) Let X_k denote the outcome of the kth spinn. Describe the σ-algebra generated by X_k in terms of a suitable partition of the sample space.

Problem 2. Show that $X(\omega) = \omega^3 + \omega$ defines a random variable on the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$ and $\mathbb{P}$ is some probability measure.

Problem 3. An urn contains initially one red and one blue ball, and in each round a ball is drawn uniformly at random from the urn. If the drawn ball is blue, then the ball is replaced together with an additional two blue balls. If the drawn ball is red, then the ball is returned along with an additional ball of each colour. Show that red will be drawn infinitely many times, almost surely.

Problem 4. Let $X_1, X_2, \dots$ be independent random variables taking values ± 1 with equal probability. Let $Y_n = \sum_{k=1}^{n} \frac{1}{k} X_k$. Show that the limit $\lim_{n\to\infty} Y_n$ exists almost surely.

Part II

Problem 5. An urn contains initially one red and one blue ball. Perform twice the following operation: Draw a ball from the urn and replace the ball together with another ball of the same colour.

(a) Construct a probability space corresponding to the above experiment.

(b) Let A denote the event that the first draw is red, and B the event that exactly one draw gives red. Are A and B independent?

(c) Let X denote the number of red balls drawn and let $Y = \mathbf{1}_A$ denote the indicator variable of A. Are X and Y independent?

Problem 6. Consider $([0,1], \mathcal{B}[0,1], \mathbb{P})$ where $\mathbb{P}$ denotes Lebesgue measure. Let $X(\omega) = \mathbf{1}_{[0,1/2]}(\omega)$ and $Y(\omega) = \omega$.

(a) Determine $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(b) Determine $\mathbb{P}(Y \leq y|X)$ for all $y \in [0,1]$.

Problem 7. Let $X_1, X_2, \ldots$ be independent random variables taking values ± 1 with equal probability. Let $S_0 = a$, $S_n = a + X_1 + \ldots + X_n$ for $n \geq 1$, and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Let

$$T = \min\{n \geq 1 : X_n = X_{n-1} = 1\},$$
$$U = \min\{n \geq 1 : X_n = X_{n-1} = -1\},$$

and set $V = \min\{T, U\}$.

(a) Show that $(S_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$.

(b) Determine $\mathbb{E}[S_T]$, $\mathbb{E}[S_U]$ and $\mathbb{E}[S_V]$.

Problem 8. Consider an urn containing one red and one blue ball. In each round a ball is drawn and replaced to the urn three consecutive times, and thereafter a ball of the colour obtained the majority of times (at least two out of the three draws) is added to the urn. That is, in each round three balls are drawn with replacement, and the colour obtained the majority of times is reinforced. Let Y_n denote the proportion of red balls in the urn after n rounds, and let $f(x)$ denote the probability that a red ball is added when the current proportion of red balls equals x.

(a) Show that $f(x) = 3x^2 - 2x^3$.

(b) Show that $(Y_n)_{n\geq 0}$ is a stochastic approximation process, i.e. satisfies

$$Y_{n+1} - Y_n = \frac{1}{n+3}[F(Y_n) + \xi_{n+1}],$$

where $F(x) = f(x) - x$ and $\mathbb{E}[\xi_{n+1}|Y_1, \ldots, Y_n] = 0$.

(c) Show that $(Y_n)_{n\geq 0}$ converges almost surely, and determine the limiting distribution.