Exam Probability theory III 2024-11-01

1 (a) A suitable construction is to represent the different outcomes by the numbers 0 to 36, and give any quadruple of outcomes equal probability.

$$\Omega = \{0,1,2,\ldots,36\}^4$$

$$\mathcal{F} = \text{power set of } \Omega$$

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|} = \frac{1}{37^4} \quad \text{for every } \omega \in \Omega.$$

2p

(b) For $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega$ and $k = 1,2,3,4$ we have

$$X_k(\omega) = \omega_k.$$

Hence X_k partitions Ω into 37 sets $A_0, A_1, \ldots, A_{36}$ where

$$A_n = \{\omega \in \Omega : \omega_k = n\}.$$

The σ-algebra generated by X_k equals the σ-algebra generated by the partition $\mathcal{P} = \{A_0, \ldots, A_{36}\}$.

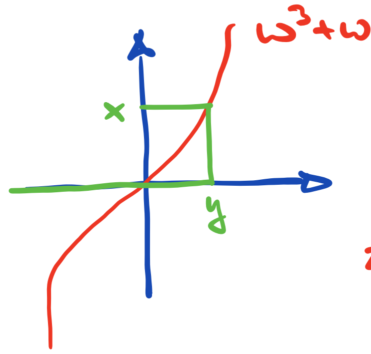
3p

2 Let $X(\omega) = \omega^3 + \omega = \omega(\omega^2+1)$

By proposition, it will suffice to show that

$$\{\omega \in \mathbb{R} : X(\omega) \leq x\} \in \mathcal{B}(\mathbb{R})$$

for all $x \in \mathbb{R}$ in order for X to be a random variable on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.



2p

Differentiation gives

$$X'(\omega) = 3\omega^2 + 1 > 0 \quad \text{for all } \omega \in \mathbb{R}.$$

X is hence strictly increasing, and for all $x \in \mathbb{R}$

$$\{\omega \in \mathbb{R} : X(\omega) \leq x\} = (-\infty, y]$$

3p

for some $y \in \mathbb{R}$. Since these sets belong to $\mathcal{B}(\mathbb{R})$, it follows that X is a random variable on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

$\boxed{3}$ Let A_k denote the event that the red ball initially in the urn is drawn in round k. There are initially two balls in the urn, and two balls are added each round. Hence

$$\mathbb{P}(A_k) = \frac{1}{2k} \quad \text{for all } k \geq 1.$$ 2p

The outcomes of previous rounds do not affect the occurrence of A_k, so the events $A_1, A_2, \ldots$ are independent. Then since

$$\sum_{k\geq 1} \mathbb{P}(A_k) = \infty,$$ 3p

the second Borel-Cantelli lemma gives $\mathbb{P}(A_k \text{ i.o.}) = 1$.

$\boxed{4}$ Let $Y_n = \sum_{k=1}^{n} \frac{1}{k} X_k$, and set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

We first show that $(Y_n)_{n\geq 1}$ is a martingale wrt $(\mathcal{F}_n)_{n\geq 1}$.

Clearly Y_n is $\mathcal{F}_n$-measurable, $\mathbb{E}[|Y_n|] \leq n$ and

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \sum_{k=1}^{n} \frac{1}{k} X_k + \frac{1}{n+1} \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = Y_n.$$ 2p

Hence it is a martingale. Moreover

$$\mathbb{E}[Y_n^2] = \sum_{k=1}^{n} \mathbb{E}[(\tfrac{1}{k} X_k)^2] = \sum_{k=1}^{n} \frac{1}{k^2} \leq \frac{\pi^2}{6}$$

for all $n \geq 1$, where we in the first step used that martingale increments are uncorrelated.

That is, $(Y_n)_{n\geq 1}$ is an L^2-bounded ($\sup \mathbb{E}[Y_n^2] < \infty$) martingale, and by the martingale convergence thm almost surely convergent. 3p

$\boxed{5}$ (a) The following corresponds to the described experiment

$$\Omega = \{rr, rb, br, bb\}$$

$$\mathcal{F} = \text{power set of } \Omega$$

$$\mathbb{P}(rr) = \tfrac{1}{2}\cdot\tfrac{2}{3} \quad \mathbb{P}(rb) = \tfrac{1}{2}\cdot\tfrac{1}{3} \quad \mathbb{P}(br) = \tfrac{1}{2}\cdot\tfrac{1}{3} \quad \mathbb{P}(bb) = \tfrac{1}{2}\cdot\tfrac{2}{3}$$

Above r denotes a red draw and b a blue draw. 4p

(b) In the above notation we have

$$A = \{rr, rb\}$$
$$B = \{rb, br\}$$
$$A \cap B = \{rb\}$$

Hence 3p

$$\mathbb{P}(A\cap B) = \tfrac{1}{6} \qquad \mathbb{P}(A) = \tfrac{1}{3} + \tfrac{1}{6} = \tfrac{1}{2}$$
$$\mathbb{P}(B) = \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{3}$$

Hence $\mathbb{P}(A\cap B) = \mathbb{P}(A)\mathbb{P}(B)$ and A, B <u>are</u> independent.

(c) For X, Y to be independent we need

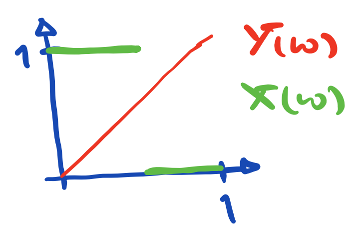
$$\mathbb{P}(X=i, Y=j) = \mathbb{P}(X=i)\,\mathbb{P}(Y=j)$$

for all $i = 0,1,2$ and $j = 0,1$. However,

$$\mathbb{P}(X=0, Y=1) = \mathbb{P}(\emptyset) = 0$$
$$\mathbb{P}(X=0) = \mathbb{P}(bb) = \tfrac{1}{3}$$
$$\mathbb{P}(Y=1) = \mathbb{P}(A) = 1/3. \qquad \text{3p}$$

Hence X and Y are <u>not</u> independent.

$\boxed{6}$ (a) X takes only two values and is thus simple.
By definition of expectation



$$\mathbb{E}[X] = 1\cdot\mathbb{P}(X=1) + 0\cdot\mathbb{P}(X=0)$$
$$= \mathbb{P}([0, 1/2]) = \tfrac{1}{2}$$

Y is not simple but bounded, and its expectation is thus approximated from below and above by that of simple functions. By dividing $[0,1]$ into intervals of length $1/n$, we obtain upper and lower bounds on $\mathbb{E}[Y]$ that coincide with Riemann sums. Hence 5p

$$\mathbb{E}[Y] = \int_0^1 w\,dw = \frac{1}{2}.$$

(b) By definition $\mathbb{P}(Y \le y \mid X)$ is the random variable which on the event $\{X=0\}$ takes the value $\mathbb{P}(Y\le y \mid X=0)$ and on $\{X=1\}$ takes the value $\mathbb{P}(Y \le y \mid X=1)$. Hence 2p

$$\mathbb{P}(Y\le y \mid X=0) = \frac{\mathbb{P}(Y\le y, X=0)}{\mathbb{P}(X=0)} = \frac{\mathbb{P}([0,y]\cap(\frac{1}{2},1])}{\mathbb{P}((1/2,1])}$$

$$= \begin{cases} 0 & y \le 1/2 \\ 2y-1 & y > 1/2 \end{cases}$$

$$\mathbb{P}(Y\le y \mid X=1) = \frac{\mathbb{P}([0,y]\cap[0,1/2])}{\mathbb{P}([0,1/2])} = \begin{cases} 2y & y\le 1/2 \\ 1 & y>1/2 \end{cases}$$

We may summarise the above as

$$\mathbb{P}(Y\le y \mid X) = \begin{cases} 2y\,\mathbb{1}\{X=1\} & y \le 1/2 \\ \mathbb{1}\{X=1\} + (2y-1)\mathbb{1}\{X=0\} & y > 1/2 \end{cases}$$ 3p

Or we may simply note that the conditional distribution of Y given X is uniform on the interval $[1/2, 1]$ on the event $\{X=0\}$ and uniform on $[0,1/2]$ on $\{X=1\}$.

$\boxed{7}$ (a) Clearly S_n is $\mathcal{F}_n$-measurable, $|S_n| \le a+n$ and so $\mathbb{E}[|S_n|]<\infty$. Moreover,

$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = a + \sum_{k=1}^n X_k + \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = S_n + 0.$$

Hence $(S_n)_{n\ge 0}$ is a martingale. 3p

(b) We want to apply optional stopping, and first argue that T, U and V are stopping times. For $n \geq 2$

$$\{T=n\} = \{X_n = X_{n-1} = 1\} \cap \Big[\bigcap_{k=2}^{n-1} \{X_k = -1\} \cup \{X_{k-1} = -1\} \Big],$$

so the event $\{T=n\}$ can be written in terms of sets in $\mathcal{F}_n$, and is then itself in $\mathcal{F}_n$. The argument for U is analogous.

For V we have, for $n \geq 2$,

$$\{V=n\} = \big[\{T=n\} \cap \{U \geq n\} \big] \cup \big[\{T \geq n\} \cap \{U=n\} \big].$$

Since

$$\{U \geq n\} = \{U < n\}^c = \Big[\bigcup_{k=2}^{n-1} \{U=k\} \Big]^c \in \mathcal{F}_n$$

3p

it follows that V is a stopping time too.

Since $|S_{n+1} - S_n| = 1$ for all n, it follows from Optional stopping III that

$$\mathbb{E}[S_T] = \mathbb{E}[S_U] = \mathbb{E}[S_V] = \mathbb{E}[S_0] = a,$$

2p

assuming $\mathbb{E}[T] < \infty$, $\mathbb{E}[U] < \infty$ and $\mathbb{E}[V] < \infty$.

By symmetry $\mathbb{E}[T] = \mathbb{E}[U]$ and $\mathbb{E}[V] \leq \mathbb{E}[T]$, so it will suffice to consider T. Let

$$A_k = \{X_{2k} = X_{2k-1} = 1\}$$

and set $N = \min\{k \geq 1 : A_k \text{ occurs}\}$. Then $T \leq 2N$, 2p
the events $A_1, A_2, \ldots$ are independent, and N thus geometrically distributed with parameter

$$\mathbb{P}(A_k) = \tfrac{1}{2} \cdot \tfrac{1}{2} = 1/4.$$

Hence $\mathbb{E}[T] \leq 2\,\mathbb{E}[N] = 8$.

$\boxed{8}$ (a) In each round three balls are drawn with replacement and the colour obtained the majority of times is reinforced.

Let Z_{n+1} denote the number of red balls drawn in round $n+1$. Then, given $Y_n = x$ we have $Z_{n+1} \sim \text{Bin}(3,x)$, and

$$\begin{aligned} f(x) &= \mathbb{P}(\text{add red} \mid Y_n = x) \\ &= \mathbb{P}(Z_{n+1} \geq 2 \mid Y_n = x) \\ &= \mathbb{P}(Z_{n+1} = 2 \mid Y_n = x) + \mathbb{P}(Z_{n+1} = 3 \mid Y_n = x) \\ &= 3 \cdot x^2(1-x) + x^3 = 3x^2 - 2x^3 \end{aligned}$$

3p

(b) For $n \geq 1$ let

$$X_n = \begin{cases} 1 & \text{red added round } n \\ 0 & \text{o/w} \end{cases}$$

We then have

$$\begin{aligned} Y_{n+1} - Y_n &= \frac{1 + \sum_{k=1}^{n+1} X_k}{n+3} - \frac{1 + \sum_{k=1}^{n} X_k}{n+2} \\ &= \frac{1}{n+3}\left[1 + \sum_{k=1}^{n+1} X_k - \frac{n+3}{n+2}\left[1 + \sum_{k=1}^{n} X_k\right]\right] \\ &= \frac{1}{n+3}\left[X_{n+1} - Y_n\right] \\ &= \frac{1}{n+3}\left[\underbrace{f(Y_n) - Y_n}_{F(Y_n)} + \underbrace{X_{n+1} - f(Y_n)}_{\xi_{n+1}}\right] \end{aligned}$$

3p

Since $\mathbb{E}[X_{n+1} - f(Y_n) \mid Y_n] = 0$, we have verified that $(Y_n)_{n \geq 0}$ is a stochastic approx process.

(c) From the stochastic approx convergence thm we know that $(Y_n)_{n \geq 0}$ is almost surely convergent. Moreover, the limit Y_∞ takes values in $\{x \in [0,1] : F(x) = 0\}$. We have

$$\begin{aligned} F(x) = f(x) - x &= 3x^2 - 2x^3 - x = -x(2x^2 - 3x + 1) \\ &= -2x(x-1)(x-\tfrac{1}{2}) \end{aligned}$$

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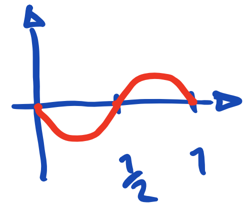

Hence $F(x)=0$ has solutions

$$x=0 \qquad x=\tfrac{1}{2} \qquad x=1.$$

Note further that F is negative on $(0,\tfrac{1}{2})$ and positive on $(\tfrac{1}{2},1)$.

Hence $x=\tfrac{1}{2}$ is an unstable equilibrium, and we have $\mathbb{P}(Y_\infty=\tfrac{1}{2})=0$.

(Note that for Y_n close to $\tfrac{1}{2}$ we have $f(Y_n)\approx \tfrac{1}{2}$ and

$$X_{n+1}-f(Y_n)\approx \pm \tfrac{1}{2}.$$

Hence the relevant condition is fulfilled.)

2p

We may conclude that $Y_\infty \in \{0,1\}$. By symmetry it must take both values with equal probability.