MT7047 – Probability theory III – exam

Date Thursday December 5, 2024
Examiner Daniel Ahlberg
Tools None.
Grading criteria The exam consists of two parts, which consist of 20 and 40 points respectively. To pass the exam a score of 14 or higher is required on Part I. If attained, then also Part II is graded, and the score on this part determines the grade. Grades are determined according to the following table:

	A	B	C	D	E
Part I	14	14	14	14	14
Part II	32	24	16	8	0

Problems of Part I may give up to five points each, and problems of Part II may give up to ten points each. Complete and well motivated solutions are required for full score. Partial solution may be rewarded with a partial score.

Part I

Problem 1. Consider a simple random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}$, where $S_0 = 0$ and which in each step jumps up with probability $p > 1/2$ and down with probability $1-p$. Determine which of the following that are stopping times with respect to $(\mathcal{F}_n)_{n\geq 1}$, where $\mathcal{F}_n = \sigma(S_1, S_2, \dots, S_n)$.

$$\begin{aligned}
T &= \min\{n \geq 1 : S_n = -1\},\\
U &= \max\{n \geq 1 : S_n = -1\},\\
V &= \min\{n \geq 1 : S_n = S_{n-k} + k\},
\end{aligned}$$

where k is a positive integer.

Problem 2. Let Ω be some set, $\mathcal{F}$ some σ-algebra on Ω and $\mathbb{P}$ and $\mathbb{P}'$ two probability measures on $(\Omega, \mathcal{F})$. For every $A \in \mathcal{F}$ let

$$\mathbb{Q}(A) := \frac{1}{2}\mathbb{P}(A) + \frac{1}{2}\mathbb{P}'(A).$$

Show that $\mathbb{Q}$ is again a probability measure on $(\Omega, \mathcal{F})$.

Problem 3. Consider a simple symmetric random walk on $\mathbb{Z}$. That is, set $S_0 = 0$ and let $S_n = \sum_{k=1}^{n} X_k$ for $n \geq 1$, where $X_1, X_2, \dots$ are independent taking values ± 1 with equal probability. Show that with probability one the random walk eventually leaves the interval $[-m, m]$, where $m \geq 1$ is an integer.

Problem 4. Let $(X_n)_{n\geq 1}$ be a bounded sequence of random variables, i.e. satisfying $\mathbb{P}(|X_n| \leq K) = 1$ for some $K < \infty$ and all $n \geq 1$. Suppose that $X_n \to X$ in probability, as $n \to \infty$, and show that $\mathbb{P}(|X| > K + 1) = 0$.

Part II

Problem 5. An infinite sequence of experiments are carried out independently, where the nth experiment is successful with probability $n^{-\alpha}$, for some $\alpha \in (0,1)$. Show that, almost surely, the sequence of experiments will result in m consecutive successes infinitely often for $m \leq 1/\alpha$, but not for $m > 1/\alpha$.

Problem 6. Consider a simple symmetric random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}$ with $S_0 = 0$. Let $a \geq 1$ be a fixed integer and set

$$T = \min\{n \geq 1 : S_n \geq 2a - n\}.$$

(a) Show that T is a stopping time.

(b) Verify that $\mathbb{E}[T] < \infty$.

(c) Deduce that $\mathbb{E}[T] = 2a$.

Problem 7. Consider $([0,1), \mathcal{B}[0,1), \mathbb{P})$, where $\mathcal{B}[0,1)$ denotes the Borel σ-algebra and $\mathbb{P}$ Lebesgue measure. Let $X(\omega) = k$ for $\omega \in [\frac{k}{10}, \frac{k+1}{10})$, for each $k = 0, 1, \ldots, 9$.

(a) Determine the σ-algebra generated by X and whether X is a random variable on the given probability space.

(b) Let $Y(\omega) = \omega$ and determine the conditional expectation $\mathbb{E}[Y|X]$.

Problem 8. An urn contains initially r red and b blue balls. In each round a ball is drawn uniformly at random, and returned to the urn together with two additional balls of the same colour.

(a) Show that the proportion of red balls in the urn has an almost sure limit.

(b) Suppose instead that the replacement rule is changed, so that one additional ball of the same colour is added and a ball of the opposite colour is removed. Does the proportion of red balls have an almost sure limit?